Last name _____

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LARSON—MATH 391—CLASSROOM WORKSHEET 15
Regular Graphs.

Facts about Symmetric $n \times n$ Matrices

- They have real eigenvalues.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- There is a basis of $\mathbb{R}^n$ consisting of orthogonal eigenvectors of the matrix..

The **Rayleigh-Ritz** or **Min-Max Theorem**: For any real symmetric matrix A with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$,

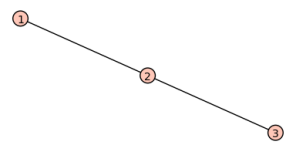
$$\lambda_n \leq \frac{(\vec{x}, A\vec{x})}{(\vec{x}, \vec{x})} \leq \lambda_1,$$

with equality in the upper bound by any eigenvector corresponding to λ_1 and in the lower bound with any eigenvector corresponding to λ_n.

Theorem. The average degree of a graph is no more than its largest eigenvalue.

Regular Graphs

1. We already showed that the row-sums of the adjacency matrix are the vertex degrees. If these are all the same the graph is *regular*.
2. The Theorem above implies that the maximum degree Δ of a regular graph is no more than its largest eigenvalue.
3. The Max-Min Theorem also implies that the average of the squares of the eigenvalues of a graph are no more than its largest eigenvalue. For regular graphs with $\Delta = \lambda_1$ we then get equality!

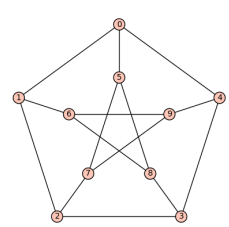


Let:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = \sqrt{2}$, $\lambda_2 = 0$, $\lambda_3 = -\sqrt{2}$.

1. Check that the row-sums are the degrees of the corresponding vertices.
2. Find A^2.
3. Check that the diagonal entries of A^2 are the degrees of the corresponding vertices.
4. Find the eigenvalues of A^2.
5. Check that the sum of the squares of the eigenvalues of A equals the sum of its degrees.
6. Check that $\frac{1}{n}\sum deg(v) \leq \lambda_1$.
7. Check that $\frac{1}{n}\sum \lambda_i^2 \leq \lambda_1$.



8. What can you say about the eigenvalues of the Petersen graph without any computation?