LARSON—MATH 353—CLASSROOM WORKSHEET 22
Random Graphs.

1. Log in to your Sage Cloud account.
   (a) Start the Chrome browser.
   (b) Go to http://cloud.sagemath.com and sign in.
   (c) You should see an existing Project for our class. Click on that.
   (d) Click “New”, call it c22, then click “Sage Worksheet”.

**Question 1:** What’s the probability that a graph is connected? We’ve seen that the order (number of vertices) is essential to the answer of this question.

**Question 2:** What’s the probability that a graph with order \( n \) is connected?

The answer is: \[
\frac{\text{The number of connected graphs of order } n}{\text{The number of graphs with order } n}
\]

**Idea 1:** We can estimate this proportion but putting all the order \( n \) graphs in a hat, choosing a random sample, and calculating the proportion of graphs in the sample that are connected.

**Note:** As \( n \) gets large the number of graphs with order \( n \) gets *very* large. Then the obvious way of calculating the number in the numerator—generate all order \( n \) graphs and check if each is connected—is a non-starter, finite, but practically impossible.

**Idea 2:** Instead of putting all the order \( n \) graphs in a hat, we can generate individual random graphs by starting with \( n \) vertices, and for each pair of vertices, flipping a coin—if the coin comes up heads, we put an edge between those vertices. So we get a choice of a graph from our hat of graphs without actually generating all of the order \( n \) graphs. Then we can check if this random graph is connected, and repeat several times.

**Question 3:** What’s the probability that a random graph with order \( n \) is connected?

The probability that there is an edge between a pair of vertices is called the edge-probability \( p \).

**Question 4:** What’s the probability that a random graph with order \( n \) and edge probability \( p \) is connected?

**Result 1:** The probability that a random graph with order 3 and edge probability \( p = \frac{1}{2} \) is connected is 0.50.

**Result 2:** The probability that a random graph with order 100 and edge probability \( p = \frac{1}{2} \) is connected is 1.0.
The *degree* of a vertex of a graph is the number of edges the vertex is adjacent to. The *minimum degree of a graph* is the minimum of the degrees of the graph (it is a graph *invariant*). Here is code to find the minimum degree:

```python
def min_degree(g):
    return min(g.degree())
```

A graph is *Dirac* if the minimum degree of the graph is at least half of the order of the graph (it is a graph *property*). Here is code to test if a graph is Dirac:

```python
def is_dirac(g):
    n = g.order()
    return min_degree(g) >= n/2
```

2. Find the (analytic) probability that a graph with order 3 is Dirac. List all the order 3 graphs. Count the number of them that are Dirac and then divide by the number of order 3 graphs.

3. Find the empirical (experimental) probability that a graph with order 3 is Dirac. Imitate our code that forms random graphs of order 3 (and edge probability $p = 0.5$) and tests whether the graph is connected (so replace the connectedness test with our Dirac test). Do at least 100 experiments.

4. Find the (analytic) probability that a graph with order 3 and edge probability $p = 0.75$ is Dirac. List all the order 3 graphs. Calculate the probability of each graph occurring and then sum up the probabilities of the Dirac ones.

5. Find the empirical (experimental) probability that a graph with order 3 and edge probability $p = 0.75$ is Dirac. Do at least 100 experiments.