An Explicit Catalan number formula & Binary Trees

Organizational Notes

1. A Zoom recording link and class notes will be sent out after each class.

2. Remember to send your answers to the classroom worksheets. Title your email with enough to help me record your “participation”.

3. Homework #2 is due today.

Review

1. We will prove that $C_n = \frac{1}{n+1} \binom{2n}{n}$. Evaluate this formula for $n = 0, 1, 2, 3$.

2. Our initial goal is to show that $C(x) = \frac{1 - \sqrt{1 - 4x}}{x}$.

The Lemma and the Explicit Formula (Sec. 1.3, 1.4)

1. **Step 1.** Find an expression for $\frac{C(x) - 1}{x}$.

2. **Step 2.** Find a formula for $(C(x))^2$. (What does the coefficient of the $x^n$ term look like?)

3. **Step 3.** Observe that $\frac{C(x) - 1}{x} = (C(x))^2$ and solve this quadratic for $C(x)$.

4. **Step 4.** There are two solutions. Expand out $\sqrt{1 - 4x}$ using Newton’s Binomial Theorem and eliminate one of them.
5. **Step 5.** Now we are ready to prove our main theorem. First find an (infinite) series expression for \( \frac{1 - \sqrt{1 - 4x}}{2x} \) using Newton’s Binomial Theorem.

6. **Step 6.** This gives us a formula for \( C_n \). What is it?

7. **Step 7.** We claim that these expressions actually equal \( \frac{1}{n + 1} \binom{2n}{n} \). How can we show it?

**Binary Trees** (Sec. 1.5)

8. What is a binary tree?

9. How many binary trees are there with 2 nodes?

10. How many binary trees are there with 3 nodes?

11. How many binary trees are there with 4 nodes?

12. Can you *conjecture* how many binary trees there are with \( n \) nodes?

13. Can you *prove* your formula?