Chapter Two: Vector Spaces

Subsection Two.I.1: Definition and Examples

Two.I.1.18 **(a)** $0 + 0x + 0x^2 + 0x^3$
(b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(c) The constant function $f(x) = 0$
(d) The constant function $f(n) = 0$

Two.I.1.19 **(a)** $3 + 2x - x^2$ **(b)** $\begin{pmatrix} -1 & +1 \\ 0 & -3 \end{pmatrix}$ **(c)** $-3e^x + 2e^{-x}$

Two.I.1.20 Most of the conditions are easy to check; use Example 1.3 as a guide. Here are some comments.
(a) This is just like Example 1.3; the zero element is $0 + 0x$.
(b) The zero element of this space is the 2×2 matrix of zeroes.
(c) The zero element is the vector of zeroes.
(d) Closure of addition involves noting that the sum

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}$$

is in L because $(x_1+x_2)+(y_1+y_2)-(z_1+z_2)+(w_1+w_2) = (x_1+y_1-z_1+w_1)+(x_2+y_2-z_2+w_2) = 0+0$. Closure of scalar multiplication is similar. Note that the zero element, the vector of zeroes, is in L.

Two.I.1.21 In each item the set is called Q. For some items, there are other correct ways to show that Q is not a vector space.
(a) It is not closed under addition; it fails to meet condition (1).

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

(b) It is not closed under addition.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

(c) It is not closed under addition.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \notin Q$$

(d) It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

(e) It is empty, violating condition (4).

Two.I.1.22 The usual operations $(v_0 + v_1 i) + (w_0 + w_1 i) = (v_0 + w_0) + (v_1 + w_1)i$ and $r(v_0 + v_1 i) = (rv_0) + (rv_1)i$ suffice. The check is easy.

Two.I.1.23 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

Two.I.1.24 The natural operations are $(v_1 x + v_2 y + v_3 z) + (w_1 x + w_2 y + w_3 z) = (v_1 + w_1)x + (v_2 + w_2)y + (v_3 + w_3)z$ and $r \cdot (v_1 x + v_2 y + v_3 z) = (rv_1)x + (rv_2)y + (rv_3)z$. The check that this is a vector space is easy; use Example 1.3 as a guide.

Two.I.1.25 The '+' operation is not commutative (that is, condition (2) is not met); producing two members of the set witnessing this assertion is easy.

Two.I.1.26 **(a)** It is not a vector space.

$$(1+1)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\neq\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}$$

(b) It is not a vector space.

$$1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\neq\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Two.I.1.27 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

(a) Yes.
(b) Yes.
(c) No, it is not closed under addition. The vector of all 1/4's, when added to itself, makes a nonmember.
(d) Yes.
(e) No, $f(x)=e^{-2x}+(1/2)$ is in the set but $2\cdot f$ is not (that is, condition (6) fails).

Two.I.1.28 It is a vector space. Most conditions of the definition of vector space are routine; we here check only closure. For addition, $(f_1+f_2)\,(7)=f_1(7)+f_2(7)=0+0=0$. For scalar multiplication, $(r\cdot f)\,(7)=rf(7)=r0=0$.

Two.I.1.29 We check Definition 1.1.

First, closure under '+' holds because the product of two positive reals is a positive real. The second condition is satisfied because real multiplication commutes. Similarly, as real multiplication associates, the third checks. For the fourth condition, observe that multiplying a number by $1\in\mathbb{R}^+$ won't change the number. Fifth, any positive real has a reciprocal that is a positive real.

The sixth, closure under '·', holds because any power of a positive real is a positive real. The seventh condition is just the rule that v^{r+s} equals the product of v^r and v^s. The eight condition says that $(vw)^r=v^rw^r$. The ninth condition asserts that $(v^r)^s=v^{rs}$. The final condition says that $v^1=v$.

Two.I.1.30 **(a)** No: $1\cdot(0,1)+1\cdot(0,1)\neq(1+1)\cdot(0,1)$.
(b) No; the same calculation as the prior answer shows a contition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1\cdot(0,1)\neq(0,1)$.

Two.I.1.31 It is not a vector space since it is not closed under addition, as $(x^2)+(1+x-x^2)$ is not in the set.

Two.I.1.32 **(a)** 6
(b) nm
(c) 3
(d) To see that the answer is 2, rewrite it as

$$\{\begin{pmatrix}a&0\\b&-a-b\end{pmatrix}\bigm| a,b\in\mathbb{R}\}$$

so that there are two parameters.

Two.I.1.33 A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '$\vec{+}$' and '$\vec{\cdot}$' subject to these conditions. Where $\vec{v},\vec{w}\in V$, (1) their *vector sum* $\vec{v}\mathbin{\vec{+}}\vec{w}$ is an element of V. If $\vec{u},\vec{v},\vec{w}\in V$ then (2) $\vec{v}\mathbin{\vec{+}}\vec{w}=\vec{w}\mathbin{\vec{+}}\vec{v}$ and (3) $(\vec{v}\mathbin{\vec{+}}\vec{w})\mathbin{\vec{+}}\vec{u}=\vec{v}\mathbin{\vec{+}}(\vec{w}\mathbin{\vec{+}}\vec{u})$. (4) There is a *zero vector* $\vec{0}\in V$ such that $\vec{v}\mathbin{\vec{+}}\vec{0}=\vec{v}$ for all $\vec{v}\in V$. (5) Each $\vec{v}\in V$ has an *additive inverse* $\vec{w}\in V$ such that $\vec{w}\mathbin{\vec{+}}\vec{v}=\vec{0}$. If r,s are *scalars*, that is, members of $\mathbb{R}$), and $\vec{v},\vec{w}\in V$ then (6) each *scalar multiple* $r\cdot\vec{v}$ is in V. If $r,s\in\mathbb{R}$ and $\vec{v},\vec{w}\in V$ then (7) $(r+s)\cdot\vec{v}=r\cdot\vec{v}\mathbin{\vec{+}}s\cdot\vec{v}$, and (8) $r\mathbin{\vec{\cdot}}(\vec{v}+\vec{w})=r\mathbin{\vec{\cdot}}\vec{v}+r\mathbin{\vec{\cdot}}\vec{w}$, and (9) $(rs)\mathbin{\vec{\cdot}}\vec{v}=r\mathbin{\vec{\cdot}}(s\mathbin{\vec{\cdot}}\vec{v})$, and (10) $1\mathbin{\vec{\cdot}}\vec{v}=\vec{v}$.

Two.I.1.34 **(a)** Let V be a vector space, assume that $\vec{v}\in V$, and assume that $\vec{w}\in V$ is the additive inverse of $\vec{v}$ so that $\vec{w}+\vec{v}=\vec{0}$. Because addition is commutative, $\vec{0}=\vec{w}+\vec{v}=\vec{v}+\vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.
(b) Let V be a vector space and suppose $\vec{v},\vec{s},\vec{t}\in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v}+\vec{s}=\vec{v}+\vec{t}$ gives that $-\vec{v}+\vec{v}+\vec{s}=-\vec{v}+\vec{v}+\vec{t}$, which says that $\vec{0}+\vec{s}=\vec{0}+\vec{t}$ and so $\vec{s}=\vec{t}$.

Two.I.1.35 Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v}=\vec{v}+\vec{0}=\vec{0}+\vec{v}$.

Two.I.1.36 It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

Two.I.1.37 Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

Two.I.1.38 (a) Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ are easy. The other conditions are also routine.

(b) No such set can be a vector space under the inherited operations because it does not have a zero element.

Two.I.1.39 Assume that $\vec{v} \in V$ is not $\vec{0}$.

(a) One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

(b) Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

(c) A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

(d) The solution set is either trivial, or nontrivial. In the second case, it is infinite.

Two.I.1.40 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f+g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\, f'$.

Two.I.1.41 The check is routine. Note that '1' is $1 + 0i$ and the zero elements are these.

(a) $(0 + 0i) + (0 + 0i)x + (0 + 0i)x^2$

(b) $\begin{pmatrix} 0+0i & 0+0i \\ 0+0i & 0+0i \end{pmatrix}$

Two.I.1.42 Notably absent from the definition of a vector space is a distance measure.

Two.I.1.43 (a) A small rearrangement does the trick.

$$\begin{aligned}(\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) + \vec{v}_4 &= ((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \vec{v}_4 \\ &= (\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) \\ &= \vec{v}_1 + (\vec{v}_2 + (\vec{v}_3 + \vec{v}_4)) \\ &= \vec{v}_1 + ((\vec{v}_2 + \vec{v}_3) + \vec{v}_4)\end{aligned}$$

Each equality above follows from the associativity of three vectors that is given as a condition in the definition of a vector space. For instance, the second '=' applies the rule $(\vec{w}_1 + \vec{w}_2) + \vec{w}_3 = \vec{w}_1 + (\vec{w}_2 + \vec{w}_3)$ by taking $\vec{w}_1$ to be $\vec{v}_1 + \vec{v}_2$, taking $\vec{w}_2$ to be $\vec{v}_3$, and taking $\vec{w}_3$ to be $\vec{v}_4$.

(b) The base case for induction is the three vector case. This case $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$ is required of any triple of vectors by the definition of a vector space.

For the inductive step, assume that any two sums of three vectors, any two sums of four vectors, ..., any two sums of k vectors are equal no matter how the sums are parenthesized. We will show that any sum of $k + 1$ vectors equals this one $((\cdots((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \cdots) + \vec{v}_k) + \vec{v}_{k+1}$.

Any parenthesized sum has an outermost '+'. Assume that it lies between $\vec{v}_m$ and $\vec{v}_{m+1}$ so the sum looks like this.

$$(\cdots\, \vec{v}_1 \cdots \vec{v}_m\, \cdots) + (\cdots\, \vec{v}_{m+1} \cdots \vec{v}_{k+1}\, \cdots)$$

The second half involves fewer than $k + 1$ additions, so by the inductive hypothesis we can re-parenthesize it so that it reads left to right from the inside out, and in particular, so that its outermost '+' occurs right before $\vec{v}_{k+1}$.

$$= (\cdots\, \vec{v}_1\, \cdots\, \vec{v}_m\, \cdots) + ((\cdots(\vec{v}_{m+1} + \vec{v}_{m+2}) + \cdots + \vec{v}_k) + \vec{v}_{k+1})$$

Apply the associativity of the sum of three things

$$= ((\cdots\, \vec{v}_1\, \cdots\, \vec{v}_m\, \cdots) + (\cdots\, (\vec{v}_{m+1} + \vec{v}_{m+2}) + \cdots\, \vec{v}_k)) + \vec{v}_{k+1}$$

and finish by applying the inductive hypothesis inside these outermost parenthesis.

Two.I.1.44 **(a)** We outline the check of the conditions from Definition 1.1.

Additive closure holds because if $a_0 + a_1 + a_2 = 0$ and $b_0 + b_1 + b_2 = 0$ then

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

is in the set since $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) = (a_0 + a_1 + a_2) + (b_0 + b_1 + b_2)$ is zero. The second through fifth conditions are easy.

Closure under scalar multiplication holds because if $a_0 + a_1 + a_2 = 0$ then

$$r \cdot (a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

is in the set as $ra_0 + ra_1 + ra_2 = r(a_0 + a_1 + a_2)$ is zero. The remaining conditions here are also easy.

(b) This is similar to the prior answer.

(c) Call the vector space V. We have two implications: left to right, if S is a subspace then it is closed under linear combinations of pairs of vectors and, right to left, if a nonempty subset is closed under linear combinations of pairs of vectors then it is a subspace. The left to right implication is easy; we here sketch the other one by assuming S is nonempty and closed, and checking the conditions of Definition 1.1.

First, to show closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$ as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, suppose that $\vec{s}$ is in the nonempty set S and note that $0 \cdot \vec{s} = \vec{0} \in S$; showing that the $\vec{0}$ of V acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$ closure under linear combinations shows that the vector $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is in S; showing that it is the additive inverse of $\vec{s}$ under the inherited operations is routine.

The proofs for the remaining conditions are similar.

Subsection Two.I.2: Subspaces and Spanning Sets

Two.I.2.20 By Lemma 2.9, to see if each subset of $\mathcal{M}_{2\times2}$ is a subspace, we need only check if it is nonempty and closed.

(a) Yes, it is easily checked to be nonempty and closed. This is a parametrization.

$$\{a\begin{pmatrix}1&0\\0&0\end{pmatrix} + b\begin{pmatrix}0&0\\0&1\end{pmatrix} \mid a, b \in \mathbb{R}\}$$

By the way, the parametrization also shows that it is a subspace, it is given as the span of the two-matrix set, and any span is a subspace.

(b) Yes; it is easily checked to be nonempty and closed. Alternatively, as mentioned in the prior answer, the existence of a parametrization shows that it is a subspace. For the parametrization, the condition $a + b = 0$ can be rewritten as $a = -b$. Then we have this.

$$\{\begin{pmatrix}-b&0\\0&b\end{pmatrix} \mid b \in \mathbb{R}\} = \{b\begin{pmatrix}-1&0\\0&1\end{pmatrix} \mid b \in \mathbb{R}\}$$

(c) No. It is not closed under addition. For instance,

$$\begin{pmatrix}5&0\\0&0\end{pmatrix} + \begin{pmatrix}5&0\\0&0\end{pmatrix} = \begin{pmatrix}10&0\\0&0\end{pmatrix}$$

is not in the set. (This set is also not closed under scalar multiplication, for instance, it does not contain the zero matrix.)

(d) Yes.

$$\{b\begin{pmatrix}-1&0\\0&1\end{pmatrix} + c\begin{pmatrix}0&1\\0&0\end{pmatrix} \mid b, c \in \mathbb{R}\}$$

Two.I.2.21 No, it is not closed. In particular, it is not closed under scalar multiplication because it does not contain the zero polynomial.

Two.I.2.22 **(a)** Yes, solving the linear system arising from

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix} + r_2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$$

gives $r_1 = 2$ and $r_2 = 1$.

(b) Yes; the linear system arising from $r_1(x^2)+r_2(2x+x^2)+r_3(x+x^3)=x-x^3$

$$\begin{aligned} 2r_2+r_3&=1\\ r_1+r_2\quad&=0\\ r_3&=-1\end{aligned}$$

gives that $-1(x^2)+1(2x+x^2)-1(x+x^3)=x-x^3$.

(c) No; any combination of the two given matrices has a zero in the upper right.

Two.I.2.23 **(a)** Yes; it is in that span since $1\cdot\cos^2x+1\cdot\sin^2x=f(x)$.

(b) No, since $r_1\cos^2x+r_2\sin^2x=3+x^2$ has no scalar solutions that work for all x. For instance, setting x to be 0 and π gives the two equations $r_1\cdot1+r_2\cdot0=3$ and $r_1\cdot1+r_2\cdot0=3+\pi^2$, which are not consistent with each other.

(c) No; consider what happens on setting x to be $\pi/2$ and $3\pi/2$.

(d) Yes, $\cos(2x)=1\cdot\cos^2(x)-1\cdot\sin^2(x)$.

Two.I.2.24 **(a)** Yes, for any $x,y,z\in\mathbb{R}$ this equation

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix}+r_2\begin{pmatrix}0\\2\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

has the solution $r_1=x$, $r_2=y/2$, and $r_3=z/3$.

(b) Yes, the equation

$$r_1\begin{pmatrix}2\\0\\1\end{pmatrix}+r_2\begin{pmatrix}1\\1\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives rise to this

$$\begin{aligned}2r_1+r_2\quad&=x\\ r_2\quad&=y\\ r_1\quad+r_3&=z\end{aligned}\;\xrightarrow{-(1/2)\rho_1+\rho_3}\;\xrightarrow{(1/2)\rho_2+\rho_3}\;\begin{aligned}2r_1+r_2\quad&=x\\ r_2\quad&=y\\ r_3&=-(1/2)x+(1/2)y+z\end{aligned}$$

so that, given any x, y, and z, we can compute that $r_3=(-1/2)x+(1/2)y+z$, $r_2=y$, and $r_1=(1/2)x-(1/2)y$.

(c) No. In particular, the vector

$$\begin{pmatrix}0\\0\\1\end{pmatrix}$$

cannot be gotten as a linear combination since the two given vectors both have a third component of zero.

(d) Yes. The equation

$$r_1\begin{pmatrix}1\\0\\1\end{pmatrix}+r_2\begin{pmatrix}3\\1\\0\end{pmatrix}+r_3\begin{pmatrix}-1\\0\\0\end{pmatrix}+r_4\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left(\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\1&0&0&5&z\end{array}\right)\xrightarrow{-\rho_1+\rho_3}\xrightarrow{3\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\0&0&1&6&-x+3y+z\end{array}\right)$$

We have infinitely many solutions. We can, for example, set r_4 to be zero and solve for r_3, r_2, and r_1 in terms of x, y, and z by the usual methods of back-substitution.

(e) No. The equation

$$r_1\begin{pmatrix}2\\1\\1\end{pmatrix}+r_2\begin{pmatrix}3\\0\\1\end{pmatrix}+r_3\begin{pmatrix}5\\1\\2\end{pmatrix}+r_4\begin{pmatrix}6\\0\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left(\begin{array}{cccc|c}2&3&5&6&x\\1&0&1&0&y\\1&1&2&2&z\end{array}\right)\xrightarrow[-(1/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-(1/3)\rho_2+\rho_3}\left(\begin{array}{cccc|c}2&3&5&6&x\\0&-3/2&-3/2&-3&-(1/2)x+y\\0&0&0&0&-(1/3)x-(1/3)y+z\end{array}\right)$$

This shows that not every three-tall vector can be so expressed. Only the vectors satisfying the restriction that $-(1/3)x-(1/3)y+z=0$ are in the span. (To see that any such vector is indeed expressible, take r_3 and r_4 to be zero and solve for r_1 and r_2 in terms of x, y, and z by back-substitution.)

Two.I.2.25 **(a)** $\{\begin{pmatrix}c & b & c\end{pmatrix} \mid b,c\in\mathbb{R}\} = \{b\begin{pmatrix}0 & 1 & 0\end{pmatrix}+c\begin{pmatrix}1 & 0 & 1\end{pmatrix} \mid b,c\in\mathbb{R}\}$ The obvious choice for the set that spans is $\{\begin{pmatrix}0 & 1 & 0\end{pmatrix},\begin{pmatrix}1 & 0 & 1\end{pmatrix}\}$.

(b) $\{\begin{pmatrix}-d & b\\ c & d\end{pmatrix} \mid b,c,d\in\mathbb{R}\} = \{b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+d\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \mid b,c,d\in\mathbb{R}\}$ One set that spans this space consists of those three matrices.

(c) The system

$$\begin{aligned} a+3b \qquad\quad &= 0\\ 2a \qquad -c-d &= 0 \end{aligned}$$

gives $b=-(c+d)/6$ and $a=(c+d)/2$. So one description is this.

$$\{c\begin{pmatrix}1/2 & -1/6\\ 1 & 0\end{pmatrix}+d\begin{pmatrix}1/2 & -1/6\\ 0 & 1\end{pmatrix} \mid c,d\in\mathbb{R}\}$$

That shows that a set spanning this subspace consists of those two matrices.

(d) The $a=2b-c$ gives $\{(2b-c)+bx+cx^3 \mid b,c\in\mathbb{R}\} = \{b(2+x)+c(-1+x^3) \mid b,c\in\mathbb{R}\}$. So the subspace is the span of the set $\{2+x,-1+x^3\}$.

(e) The set $\{a+bx+cx^2 \mid a+7b+49c=0\}$ parametrized as $\{b(-7+x)+c(-49+x^2) \mid b,c\in\mathbb{R}\}$ has the spanning set $\{-7+x,-49+x^2\}$.

Two.I.2.26 Each answer given is only one out of many possible.

(a) We can parametrize in this way

$$\{\begin{pmatrix}x\\ 0\\ z\end{pmatrix} \mid x,z\in\mathbb{R}\} = \{x\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}+z\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \mid x,z\in\mathbb{R}\}$$

giving this for a spanning set.

$$\{\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\}$$

(b) Parametrize it with $\{y\begin{pmatrix}-2/3\\ 1\\ 0\end{pmatrix}+z\begin{pmatrix}-1/3\\ 0\\ 1\end{pmatrix} \mid y,z\in\mathbb{R}\}$ to get $\{\begin{pmatrix}-2/3\\ 1\\ 0\end{pmatrix},\begin{pmatrix}-1/3\\ 0\\ 1\end{pmatrix}\}$.

(c) $\{\begin{pmatrix}1\\ -2\\ 1\\ 0\end{pmatrix},\begin{pmatrix}-1/2\\ 0\\ 0\\ 1\end{pmatrix}\}$

(d) Parametrize the description as $\{-a_1+a_1x+a_3x^2+a_3x^3 \mid a_1,a_3\in\mathbb{R}\}$ to get $\{-1+x,x^2+x^3\}$.

(e) $\{1,x,x^2,x^3,x^4\}$

(f) $\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}$

Two.I.2.27 Technically, no. Subspaces of $\mathbb{R}^3$ are sets of three-tall vectors, while $\mathbb{R}^2$ is a set of two-tall vectors. Clearly though, $\mathbb{R}^2$ is "just like" this subspace of $\mathbb{R}^3$.

$$\{\begin{pmatrix}x\\ y\\ 0\end{pmatrix} \mid x,y\in\mathbb{R}\}$$

Two.I.2.28 Of course, the addition and scalar multiplication operations are the ones inherited from the enclosing space.

(a) This is a subspace. It is not empty as it contains at least the two example functions given. It is closed because if f_1,f_2 are even and c_1,c_2 are scalars then we have this.

$$(c_1f_1+c_2f_2)(-x) = c_1\,f_1(-x)+c_2\,f_2(-x) = c_1\,f_1(x)+c_2\,f_2(x) = (c_1f_1+c_2f_2)(x)$$

(b) This is also a subspace; the check is similar to the prior one.

Two.I.2.29 It can be improper. If $\vec{v}=\vec{0}$ then this is a trivial subspace. At the opposite extreme, if the vector space is $\mathbb{R}^1$ and $\vec{v}\neq\vec{0}$ then the subspace is all of $\mathbb{R}^1$.

Two.I.2.30 No, such a set is not closed. For one thing, it does not contain the zero vector.

Two.I.2.31 No. The only subspaces of $\mathbb{R}^1$ are the space itself and its trivial subspace. Any subspace S of $\mathbb{R}$ that contains a nonzero member $\vec{v}$ must contain the set of all of its scalar multiples $\{r\cdot\vec{v} \mid r\in\mathbb{R}\}$. But this set is all of $\mathbb{R}$.

Two.I.2.32 Item (1) is checked in the text.

Item (2) has five conditions. First, for closure, if $c \in \mathbb{R}$ and $\vec{s} \in S$ then $c \cdot \vec{s} \in S$ as $c \cdot \vec{s} = c \cdot \vec{s} + 0 \cdot \vec{0}$. Second, because the operations in S are inherited from V, for $c, d \in \mathbb{R}$ and $\vec{s} \in S$, the scalar product $(c+d) \cdot \vec{s}$ in S equals the product $(c+d) \cdot \vec{s}$ in V, and that equals $c \cdot \vec{s} + d \cdot \vec{s}$ in V, which equals $c \cdot \vec{s} + d \cdot \vec{s}$ in S.

The check for the third, fourth, and fifth conditions are similar to the second conditions's check just given.

Two.I.2.33 An exercise in the prior subsection shows that every vector space has only one zero vector (that is, there is only one vector that is the additive identity element of the space). But a trivial space has only one element and that element must be this (unique) zero vector.

Two.I.2.34 As the hint suggests, the basic reason is the Linear Combination Lemma from the first chapter. For the full proof, we will show mutual containment between the two sets.

The first containment $[[S]] \supseteq [S]$ is an instance of the more general, and obvious, fact that for any subset T of a vector space, $[T] \supseteq T$.

For the other containment, that $[[S]] \subseteq [S]$, take m vectors from $[S]$, namely $c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1}$, $\ldots$, $c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m}$, and note that any linear combination of those

$$r_1(c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1})+\cdots+r_m(c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m})$$

is a linear combination of elements of S

$$=(r_1c_{1,1})\vec{s}_{1,1}+\cdots+(r_1c_{1,n_1})\vec{s}_{1,n_1}+\cdots+(r_mc_{1,m})\vec{s}_{1,m}+\cdots+(r_mc_{1,n_m})\vec{s}_{1,n_m}$$

and so is in $[S]$. That is, simply recall that a linear combination of linear combinations (of members of S) is a linear combination (again of members of S).

Two.I.2.35 **(a)** It is not a subspace because these are not the inherited operations. For one thing, in this space,

$$0 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

while this does not, of course, hold in $\mathbb{R}^3$.

(b) We can combine the argument showing closure under addition with the argument showing closure under scalar multiplication into one single argument showing closure under linear combinations of two vectors. If $r_1, r_2, x_1, x_2, y_1, y_2, z_1, z_2$ are in $\mathbb{R}$ then

$$r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + 1 \\ r_1y_1 \\ r_1z_1 \end{pmatrix} + \begin{pmatrix} r_2x_2 - r_2 + 1 \\ r_2y_2 \\ r_2z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + r_2x_2 - r_2 + 1 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}$$

(note that the definition of addition in this space is that the first components combine as $(r_1x_1 - r_1 + 1) + (r_2x_2 - r_2 + 1) - 1$, so the first component of the last vector does not say '$+2$'). Adding the three components of the last vector gives $r_1(x_1 - 1 + y_1 + z_1) + r_2(x_2 - 1 + y_2 + z_2) + 1 = r_1 \cdot 0 + r_2 \cdot 0 + 1 = 1$.

Most of the other checks of the conditions are easy (although the oddness of the operations keeps them from being routine). Commutativity of addition goes like this.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 - 1 \\ y_2 + y_1 \\ z_2 + z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Associativity of addition has

$$(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 - 1) + x_3 - 1 \\ (y_1 + y_2) + y_3 \\ (z_1 + z_2) + z_3 \end{pmatrix}$$

while

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}) = \begin{pmatrix} x_1 + (x_2 + x_3 - 1) - 1 \\ y_1 + (y_2 + y_3) \\ z_1 + (z_2 + z_3) \end{pmatrix}$$

and they are equal. The identity element with respect to this addition operation works this way

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x + 1 - 1 \\ y + 0 \\ z + 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the additive inverse is similar.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -x+2 \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} x+(-x+2)-1 \\ y-y \\ z-z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The conditions on scalar multiplication are also easy. For the first condition,

$$(r+s)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (r+s)x-(r+s)+1 \\ (r+s)y \\ (r+s)z \end{pmatrix}$$

while

$$r\begin{pmatrix} x \\ y \\ z \end{pmatrix} + s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx-r+1 \\ ry \\ rz \end{pmatrix} + \begin{pmatrix} sx-s+1 \\ sy \\ sz \end{pmatrix} = \begin{pmatrix} (rx-r+1)+(sx-s+1)-1 \\ ry+sy \\ rz+sz \end{pmatrix}$$

and the two are equal. The second condition compares

$$r\cdot(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = r\cdot\begin{pmatrix} x_1+x_2-1 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} r(x_1+x_2-1)-r+1 \\ r(y_1+y_2) \\ r(z_1+z_2) \end{pmatrix}$$

with

$$r\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} rx_1-r+1 \\ ry_1 \\ rz_1 \end{pmatrix} + \begin{pmatrix} rx_2-r+1 \\ ry_2 \\ rz_2 \end{pmatrix} = \begin{pmatrix} (rx_1-r+1)+(rx_2-r+1)-1 \\ ry_1+ry_2 \\ rz_1+rz_2 \end{pmatrix}$$

and they are equal. For the third condition,

$$(rs)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rsx-rs+1 \\ rsy \\ rsz \end{pmatrix}$$

while

$$r(s\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = r(\begin{pmatrix} sx-s+1 \\ sy \\ sz \end{pmatrix}) = \begin{pmatrix} r(sx-s+1)-r+1 \\ rsy \\ rsz \end{pmatrix}$$

and the two are equal. For scalar multiplication by 1 we have this.

$$1\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x-1+1 \\ 1y \\ 1z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus all the conditions on a vector space are met by these two operations.

Remark. A way to understand this vector space is to think of it as the plane in $\mathbb{R}^3$

$$P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bigm| x+y+z=0\}$$

displaced away from the origin by 1 along the x-axis. Then addition becomes: to add two members of this space,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

(such that $x_1+y_1+z_1=1$ and $x_2+y_2+z_2=1$) move them back by 1 to place them in P and add as usual,

$$\begin{pmatrix} x_1-1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2-1 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2-2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} \qquad \text{(in } P\text{)}$$

and then move the result back out by 1 along the x-axis.

$$\begin{pmatrix} x_1+x_2-1 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}.$$

Scalar multiplication is similar.

(c) For the subspace to be closed under the inherited scalar multiplication, where $\vec{v}$ is a member of that subspace,

$$0 \cdot \vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

must also be a member.

The converse does not hold. Here is a subset of $\mathbb{R}^3$ that contains the origin

$$\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$$

(this subset has only two elements) but is not a subspace.

Two.I.2.36 **(a)** $(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) - (\vec{v}_1 + \vec{v}_2) = \vec{v}_3$
(b) $(\vec{v}_1 + \vec{v}_2) - (\vec{v}_1) = \vec{v}_2$
(c) Surely, $\vec{v}_1$.
(d) Taking the one-long sum and subtracting gives $(\vec{v}_1) - \vec{v}_1 = \vec{0}$.

Two.I.2.37 Yes; any space is a subspace of itself, so each space contains the other.

Two.I.2.38 **(a)** The union of the x-axis and the y-axis in $\mathbb{R}^2$ is one.
(b) The set of integers, as a subset of $\mathbb{R}^1$, is one.
(c) The subset $\{\vec{v}\}$ of $\mathbb{R}^2$ is one, where $\vec{v}$ is any nonzero vector.

Two.I.2.39 Because vector space addition is commutative, a reordering of summands leaves a linear combination unchanged.

Two.I.2.40 We always consider that span in the context of an enclosing space.

Two.I.2.41 It is both 'if' and 'only if'.

For 'if', let S be a subset of a vector space V and assume $\vec{v} \in S$ satisfies $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ where $c_1, \ldots, c_n$ are scalars and $\vec{s}_1, \ldots, \vec{s}_n \in S$. We must show that $[S \cup \{\vec{v}\}] = [S]$.

Containment one way, $[S] \subseteq [S \cup \{\vec{v}\}]$ is obvious. For the other direction, $[S \cup \{\vec{v}\}] \subseteq [S]$, note that if a vector is in the set on the left then it has the form $d_0\vec{v} + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ where the d's are scalars and the $\vec{t}$'s are in S. Rewrite that as $d_0(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ and note that the result is a member of the span of S.

The 'only if' is clearly true — adding $\vec{v}$ enlarges the span to include at least $\vec{v}$.

Two.I.2.42 **(a)** Always.

Assume that A, B are subspaces of V. Note that their intersection is not empty as both contain the zero vector. If $\vec{w}, \vec{s} \in A \cap B$ and r, s are scalars then $r\vec{v} + s\vec{w} \in A$ because each vector is in A and so a linear combination is in A, and $r\vec{v} + s\vec{w} \in B$ for the same reason. Thus the intersection is closed. Now Lemma 2.9 applies.

(b) Sometimes (more precisely, only if $A \subseteq B$ or $B \subseteq A$).

To see the answer is not 'always', take V to be $\mathbb{R}^3$, take A to be the x-axis, and B to be the y-axis. Note that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in A \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in B \quad \text{but} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin A \cup B$$

as the sum is in neither A nor B.

The answer is not 'never' because if $A \subseteq B$ or $B \subseteq A$ then clearly $A \cup B$ is a subspace.

To show that $A \cup B$ is a subspace only if one subspace contains the other, we assume that $A \not\subseteq B$ and $B \not\subseteq A$ and prove that the union is not a subspace. The assumption that A is not a subset of B means that there is an $\vec{a} \in A$ with $\vec{a} \notin B$. The other assumption gives a $\vec{b} \in B$ with $\vec{b} \notin A$. Consider $\vec{a} + \vec{b}$. Note that sum is not an element of A or else $(\vec{a} + \vec{b}) - \vec{a}$ would be in A, which it is not. Similarly the sum is not an element of B. Hence the sum is not an element of $A \cup B$, and so the union is not a subspace.

(c) Never. As A is a subspace, it contains the zero vector, and therefore the set that is A's complement does not. Without the zero vector, the complement cannot be a vector space.

Two.I.2.43 The span of a set does not depend on the enclosing space. A linear combination of vectors from S gives the same sum whether we regard the operations as those of W or as those of V, because the operations of W are inherited from V.

Two.I.2.44 It is; apply Lemma 2.9. (You must consider the following. Suppose B is a subspace of a vector space V and suppose $A \subseteq B \subseteq V$ is a subspace. From which space does A inherit its operations? The answer is that it doesn't matter — A will inherit the same operations in either case.)

Two.I.2.45 **(a)** Always; if $S \subseteq T$ then a linear combination of elements of S is also a linear combination of elements of T.

(b) Sometimes (more precisely, if and only if $S \subseteq T$ or $T \subseteq S$).

The answer is not 'always' as is shown by this example from $\mathbb{R}^3$

$$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}, \quad T = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

because of this.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \in [S \cup T] \qquad \begin{pmatrix}1\\1\\1\end{pmatrix} \notin [S] \cup [T]$$

The answer is not 'never' because if either set contains the other then equality is clear. We can characterize equality as happening only when either set contains the other by assuming $S \not\subseteq T$ (implying the existence of a vector $\vec{s} \in S$ with $\vec{s} \notin T$) and $T \not\subseteq S$ (giving a $\vec{t} \in T$ with $\vec{t} \notin S$), noting $\vec{s} + \vec{t} \in [S \cup T]$, and showing that $\vec{s} + \vec{t} \notin [S] \cup [T]$.

(c) Sometimes.

Clearly $[S \cap T] \subseteq [S] \cap [T]$ because any linear combination of vectors from $S \cap T$ is a combination of vectors from S and also a combination of vectors from T.

Containment the other way does not always hold. For instance, in $\mathbb{R}^2$, take

$$S = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}, \quad T = \{\begin{pmatrix}2\\0\end{pmatrix}\}$$

so that $[S] \cap [T]$ is the x-axis but $[S \cap T]$ is the trivial subspace.

Characterizing exactly when equality holds is tough. Clearly equality holds if either set contains the other, but that is not 'only if' by this example in $\mathbb{R}^3$.

$$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}, \quad T = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

(d) Never, as the span of the complement is a subspace, while the complement of the span is not (it does not contain the zero vector).

Two.I.2.46 Call the subset S. By Lemma 2.9, we need to check that $[S]$ is closed under linear combinations. If $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n, c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m \in [S]$ then for any $p, r \in \mathbb{R}$ we have

$$p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m$$

which is an element of $[S]$. (*Remark.* If the set S is empty, then that 'if ... then ...' statement is vacuously true.)

Two.I.2.47 For this to happen, one of the conditions giving the sensibleness of the addition and scalar multiplication operations must be violated. Consider $\mathbb{R}^2$ with these operations.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad r\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

The set $\mathbb{R}^2$ is closed under these operations. But it is not a vector space.

$$1 \cdot \begin{pmatrix}1\\1\end{pmatrix} \neq \begin{pmatrix}1\\1\end{pmatrix}$$

Subsection Two.II.1: Definition and Examples

Two.II.1.18 For each of these, when the subset is independent it must be proved, and when the subset is dependent an example of a dependence must be given.

(a) It is dependent. Considering

$$c_1\begin{pmatrix}1\\-3\\5\end{pmatrix}+c_2\begin{pmatrix}2\\2\\4\end{pmatrix}+c_3\begin{pmatrix}4\\-4\\14\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

gives rise to this linear system.

$$\begin{aligned}c_1+2c_2+4c_3&=0\\-3c_1+2c_2-4c_3&=0\\5c_1+4c_2+14c_3&=0\end{aligned}$$

Gauss' method

$$\left(\begin{array}{ccc|c}1&2&4&0\\-3&2&-4&0\\5&4&14&0\end{array}\right)\xrightarrow[-5\rho_1+\rho_3]{3\rho_1+\rho_2}\xrightarrow{(3/4)\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&2&4&0\\0&8&8&0\\0&0&0&0\end{array}\right)$$

yields a free variable, so there are infinitely many solutions. For an example of a particular dependence we can set c_3 to be, say, 1. Then we get $c_2=-1$ and $c_1=-2$.

(b) It is dependent. The linear system that arises here

$$\left(\begin{array}{ccc|c}1&2&3&0\\7&7&7&0\\7&7&7&0\end{array}\right)\xrightarrow[-7\rho_1+\rho_3]{-7\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&2&3&0\\0&-7&-14&0\\0&0&0&0\end{array}\right)$$

has infinitely many solutions. We can get a particular solution by taking c_3 to be, say, 1, and back-substituting to get the resulting c_2 and c_1.

(c) It is linearly independent. The system

$$\left(\begin{array}{cc|c}0&1&0\\0&0&0\\-1&4&0\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_2}\xrightarrow{\rho_3\leftrightarrow\rho_1}\left(\begin{array}{cc|c}-1&4&0\\0&1&0\\0&0&0\end{array}\right)$$

has only the solution $c_1=0$ and $c_2=0$. (We could also have gotten the answer by inspection — the second vector is obviously not a multiple of the first, and vice versa.)

(d) It is linearly dependent. The linear system

$$\left(\begin{array}{cccc|c}9&2&3&12&0\\9&0&5&12&0\\0&1&-4&-1&0\end{array}\right)$$

has more unknowns than equations, and so Gauss' method must end with at least one variable free (there can't be a contradictory equation because the system is homogeneous, and so has at least the solution of all zeroes). To exhibit a combination, we can do the reduction

$$\xrightarrow{-\rho_1+\rho_2}\xrightarrow{(1/2)\rho_2+\rho_3}\left(\begin{array}{cccc|c}9&2&3&12&0\\0&-2&2&0&0\\0&0&-3&-1&0\end{array}\right)$$

and take, say, $c_4=1$. Then we have that $c_3=-1/3$, $c_2=-1/3$, and $c_1=-31/27$.

Two.II.1.19 In the cases of independence, that must be proved. Otherwise, a specific dependence must be produced. (Of course, dependences other than the ones exhibited here are possible.)

(a) This set is independent. Setting up the relation $c_1(3-x+9x^2)+c_2(5-6x+3x^2)+c_3(1+1x-5x^2)=0+0x+0x^2$ gives a linear system

$$\left(\begin{array}{ccc|c}3&5&1&0\\-1&-6&1&0\\9&3&-5&0\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{(1/3)\rho_1+\rho_2}\xrightarrow{3\rho_2}\xrightarrow{-(12/13)\rho_2+\rho_3}\left(\begin{array}{ccc|c}3&5&1&0\\0&-13&4&0\\0&0&-128/13&0\end{array}\right)$$

with only one solution: $c_1=0$, $c_2=0$, and $c_3=0$.

(b) This set is independent. We can see this by inspection, straight from the definition of linear independence. Obviously neither is a multiple of the other.

(c) This set is linearly independent. The linear system reduces in this way

$$\left(\begin{array}{ccc|c}2&3&4&0\\1&-1&0&0\\7&2&-3&0\end{array}\right)\xrightarrow[-(7/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-(17/5)\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&3&4&0\\0&-5/2&-2&0\\0&0&-51/5&0\end{array}\right)$$

to show that there is only the solution $c_1=0$, $c_2=0$, and $c_3=0$.

(d) This set is linearly dependent. The linear system

$$\left(\begin{array}{cccc|c} 8 & 0 & 2 & 8 & 0 \\ 3 & 1 & 2 & -2 & 0 \\ 3 & 2 & 2 & 5 & 0 \end{array}\right)$$

must, after reduction, end with at least one variable free (there are more variables than equations, and there is no possibility of a contradictory equation because the system is homogeneous). We can take the free variables as parameters to describe the solution set. We can then set the parameter to a nonzero value to get a nontrivial linear relation.

Two.II.1.20 Let Z be the zero function $Z(x) = 0$, which is the additive identity in the vector space under discussion.

(a) This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$. Plugging in $x = 1$ and $x = 2$ gives a linear system

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 1 &= 0 \\ c_1 \cdot 2 + c_2 \cdot (1/2) &= 0 \end{aligned}$$

with the unique solution $c_1 = 0$, $c_2 = 0$.

(b) This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plug in $x = 0$ and $x = \pi/2$ to get

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= 0 \\ c_1 \cdot 0 + c_2 \cdot 1 &= 0 \end{aligned}$$

which obviously gives that $c_1 = 0$, $c_2 = 0$.

(c) This set is also linearly independent. Considering $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plugging in $x = 1$ and $x = e$

$$\begin{aligned} c_1 \cdot e + c_2 \cdot 0 &= 0 \\ c_1 \cdot e^e + c_2 \cdot 1 &= 0 \end{aligned}$$

gives that $c_1 = 0$ and $c_2 = 0$.

Two.II.1.21 In each case, that the set is independent must be proved, and that it is dependent must be shown by exihibiting a specific dependence.

(a) This set is dependent. The familiar relation $\sin^2(x)+\cos^2(x) = 1$ shows that $2 = c_1 \cdot (4\sin^2(x)) + c_2 \cdot (\cos^2(x))$ is satisfied by $c_1 = 1/2$ and $c_2 = 2$.

(b) This set is independent. Consider the relationship $c_1 \cdot 1 + c_2 \cdot \sin(x) + c_3 \cdot \sin(2x) = 0$ (that '0' is the zero function). Taking $x = 0$, $x = \pi/2$ and $x = \pi/4$ gives this system.

$$\begin{aligned} c_1 \phantom{+ (\sqrt{2}/2)c_2 + c_3} &= 0 \\ c_1 + \phantom{(\sqrt{2}/2)}c_2 &= 0 \\ c_1 + (\sqrt{2}/2)c_2 + c_3 &= 0 \end{aligned}$$

whose only solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

(c) By inspection, this set is independent. Any dependence $\cos(x) = c \cdot x$ is not possible since the cosine function is not a multiple of the identity function (we are applying Corollary 1.17).

(d) By inspection, we spot that there is a dependence. Because $(1 + x)^2 = x^2 + 2x + 1$, we get that $c_1 \cdot (1 + x)^2 + c_2 \cdot (x^2 + 2x) = 3$ is satisfied by $c_1 = 3$ and $c_2 = -3$.

(e) This set is dependent. The easiest way to see that is to recall the triginometric relationship $\cos^2(x) - \sin^2(x) = \cos(2x)$. (*Remark.* A person who doesn't recall this, and tries some x's, simply never gets a system leading to a unique solution, and never gets to conclude that the set is independent. Of course, this person might wonder if they simply never tried the right set of x's, but a few tries will lead most people to look instead for a dependence.)

(f) This set is dependent, because it contains the zero object in the vector space, the zero polynomial.

Two.II.1.22 No, that equation is not a linear relationship. In fact this set is independent, as the system arising from taking x to be 0, $\pi/6$ and $\pi/4$ shows.

Two.II.1.23 To emphasize that the equation $1 \cdot \vec{s} + (-1) \cdot \vec{s} = \vec{0}$ does not make the set dependent.

Two.II.1.24 We have already showed this: the Linear Combination Lemma and its corollary state that in an echelon form matrix, no nonzero row is a linear combination of the others.

Two.II.1.25 **(a)** Assume that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so that any relationship $d_0\vec{u} + d_1\vec{v} + d_2\vec{w} = \vec{0}$ leads to the conclusion that $d_0 = 0$, $d_1 = 0$, and $d_2 = 0$.

Consider the relationship $c_1(\vec{u}) + c_2(\vec{u} + \vec{v}) + c_3(\vec{u} + \vec{v} + \vec{w}) = \vec{0}$. Rewrite it to get $(c_1 + c_2 + c_3)\vec{u} + (c_2 + c_3)\vec{v} + (c_3)\vec{w} = \vec{0}$. Taking d_0 to be $c_1 + c_2 + c_3$, taking d_1 to be $c_2 + c_3$, and taking d_2

to be c_3 we have this system.

$$\begin{aligned} c_1+c_2+c_3&=0 \\ c_2+c_3&=0 \\ c_3&=0 \end{aligned}$$

Conclusion: the c's are all zero, and so the set is linearly independent.

(b) The second set is dependent

$$1\cdot(\vec{u}-\vec{v})+1\cdot(\vec{v}-\vec{w})+1\cdot(\vec{w}-\vec{u})=\vec{0}$$

whether or not the first set is independent.

Two.II.1.26 **(a)** A singleton set $\{\vec{v}\}$ is linearly independent if and only if $\vec{v}\neq\vec{0}$. For the 'if' direction, with $\vec{v}\neq\vec{0}$, we can apply Lemma 1.4 by considering the relationship $c\cdot\vec{v}=\vec{0}$ and noting that the only solution is the trivial one: $c=0$. For the 'only if' direction, just recall that Example 1.11 shows that $\{\vec{0}\}$ is linearly dependent, and so if the set $\{\vec{v}\}$ is linearly independent then $\vec{v}\neq\vec{0}$.

(*Remark.* Another answer is to say that this is the special case of Lemma 1.16 where $S=\varnothing$.)

(b) A set with two elements is linearly independent if and only if neither member is a multiple of the other (note that if one is the zero vector then it is a multiple of the other, so this case is covered). This is an equivalent statement: a set is linearly dependent if and only if one element is a multiple of the other.

The proof is easy. A set $\{\vec{v}_1,\vec{v}_2\}$ is linearly dependent if and only if there is a relationship $c_1\vec{v}_1+c_2\vec{v}_2=\vec{0}$ with either $c_1\neq 0$ or $c_2\neq 0$ (or both). That holds if and only if $\vec{v}_1=(-c_2/c_1)\vec{v}_2$ or $\vec{v}_2=(-c_1/c_2)\vec{v}_1$ (or both).

Two.II.1.27 This set is linearly dependent set because it contains the zero vector.

Two.II.1.28 The 'if' half is given by Lemma 1.14. The converse (the 'only if' statement) does not hold. An example is to consider the vector space $\mathbb{R}^2$ and these vectors.

$$\vec{x}=\begin{pmatrix}1\\0\end{pmatrix},\quad \vec{y}=\begin{pmatrix}0\\1\end{pmatrix},\quad \vec{z}=\begin{pmatrix}1\\1\end{pmatrix}$$

Two.II.1.29 **(a)** The linear system arising from

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has the unique solution $c_1=0$ and $c_2=0$.

(b) The linear system arising from

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}3\\2\\0\end{pmatrix}$$

has the unique solution $c_1=8/3$ and $c_2=-1/3$.

(c) Suppose that S is linearly independent. Suppose that we have both $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n$ and $\vec{v}=d_1\vec{t}_1+\cdots+d_m\vec{t}_m$ (where the vectors are members of S). Now,

$$c_1\vec{s}_1+\cdots+c_n\vec{s}_n=\vec{v}=d_1\vec{t}_1+\cdots+d_m\vec{t}_m$$

can be rewritten in this way.

$$c_1\vec{s}_1+\cdots+c_n\vec{s}_n-d_1\vec{t}_1-\cdots-d_m\vec{t}_m=\vec{0}$$

Possibly some of the $\vec{s}$'s equal some of the $\vec{t}$'s; we can combine the associated coefficients (i.e., if $\vec{s}_i=\vec{t}_j$ then $\cdots+c_i\vec{s}_i+\cdots-d_j\vec{t}_j-\cdots$ can be rewritten as $\cdots+(c_i-d_j)\vec{s}_i+\cdots$). That equation is a linear relationship among distinct (after the combining is done) members of the set S. We've assumed that S is linearly independent, so all of the coefficients are zero. If i is such that $\vec{s}_i$ does not equal any $\vec{t}_j$ then c_i is zero. If j is such that $\vec{t}_j$ does not equal any $\vec{s}_i$ then d_j is zero. In the final case, we have that $c_i-d_j=0$ and so $c_i=d_j$.

Therefore, the original two sums are the same, except perhaps for some $0\cdot\vec{s}_i$ or $0\cdot\vec{t}_j$ terms that we can neglect.

(d) This set is not linearly independent:

$$S=\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\}\subset\mathbb{R}^2$$

and these two linear combinations give the same result

$$\begin{pmatrix}0\\0\end{pmatrix}=2\cdot\begin{pmatrix}1\\0\end{pmatrix}-1\cdot\begin{pmatrix}2\\0\end{pmatrix}=4\cdot\begin{pmatrix}1\\0\end{pmatrix}-2\cdot\begin{pmatrix}2\\0\end{pmatrix}$$

Thus, a linearly dependent set might have indistinct sums.

In fact, this stronger statement holds: if a set is linearly dependent then it must have the property that there are two distinct linear combinations that sum to the same vector. Briefly, where $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ then multiplying both sides of the relationship by two gives another relationship. If the first relationship is nontrivial then the second is also.

Two.II.1.30 In this 'if and only if' statement, the 'if' half is clear — if the polynomial is the zero polynomial then the function that arises from the action of the polynomial must be the zero function $x \mapsto 0$. For 'only if' we write $p(x) = c_n x^n + \cdots + c_0$. Plugging in zero $p(0) = 0$ gives that $c_0 = 0$. Taking the derivative and plugging in zero $p'(0) = 0$ gives that $c_1 = 0$. Similarly we get that each c_i is zero, and p is the zero polynomial.

Two.II.1.31 The work in this section suggests that an n-dimensional non-degenerate linear surface should be defined as the span of a linearly independent set of n vectors.

Two.II.1.32 **(a)** For any $a_{1,1}, \ldots, a_{2,4}$,

$$c_1\begin{pmatrix}a_{1,1}\\a_{2,1}\end{pmatrix} + c_2\begin{pmatrix}a_{1,2}\\a_{2,2}\end{pmatrix} + c_3\begin{pmatrix}a_{1,3}\\a_{2,3}\end{pmatrix} + c_4\begin{pmatrix}a_{1,4}\\a_{2,4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

yields a linear system

$$\begin{aligned}a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 &= 0\\ a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 &= 0\end{aligned}$$

that has infinitely many solutions (Gauss' method leaves at least two variables free). Hence there are nontrivial linear relationships among the given members of $\mathbb{R}^2$.

(b) Any set five vectors is a superset of a set of four vectors, and so is linearly dependent.

With three vectors from $\mathbb{R}^2$, the argument from the prior item still applies, with the slight change that Gauss' method now only leaves at least one variable free (but that still gives infintely many solutions).

(c) The prior item shows that no three-element subset of $\mathbb{R}^2$ is independent. We know that there are two-element subsets of $\mathbb{R}^2$ that are independent — one is

$$\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$$

and so the answer is two.

Two.II.1.33 Yes; here is one.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\}$$

Two.II.1.34 Yes. The two improper subsets, the entire set and the empty subset, serve as examples.

Two.II.1.35 In $\mathbb{R}^4$ the biggest linearly independent set has four vectors. There are many examples of such sets, this is one.

$$\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}$$

To see that no set with five or more vectors can be independent, set up

$$c_1\begin{pmatrix}a_{1,1}\\a_{2,1}\\a_{3,1}\\a_{4,1}\end{pmatrix} + c_2\begin{pmatrix}a_{1,2}\\a_{2,2}\\a_{3,2}\\a_{4,2}\end{pmatrix} + c_3\begin{pmatrix}a_{1,3}\\a_{2,3}\\a_{3,3}\\a_{4,3}\end{pmatrix} + c_4\begin{pmatrix}a_{1,4}\\a_{2,4}\\a_{3,4}\\a_{4,4}\end{pmatrix} + c_5\begin{pmatrix}a_{1,5}\\a_{2,5}\\a_{3,5}\\a_{4,5}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

and note that the resulting linear system

$$\begin{aligned}a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 + a_{1,5}c_5 &= 0\\ a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 + a_{2,5}c_5 &= 0\\ a_{3,1}c_1 + a_{3,2}c_2 + a_{3,3}c_3 + a_{3,4}c_4 + a_{3,5}c_5 &= 0\\ a_{4,1}c_1 + a_{4,2}c_2 + a_{4,3}c_3 + a_{4,4}c_4 + a_{4,5}c_5 &= 0\end{aligned}$$

has four equations and five unknowns, so Gauss' method must end with at least one c variable free, so there are infinitely many solutions, and so the above linear relationship among the four-tall vectors has more solutions than just the trivial solution.

The smallest linearly independent set is the empty set.

The biggest linearly dependent set is $\mathbb{R}^4$. The smallest is $\{\vec{0}\}$.

Two.II.1.36 **(a)** The intersection of two linearly independent sets $S \cap T$ must be linearly independent as it is a subset of the linearly independent set S (as well as the linearly independent set T also, of course).

(b) The complement of a linearly independent set is linearly dependent as it contains the zero vector.

(c) We must produce an example. One, in $\mathbb{R}^2$, is

$$S = \{\begin{pmatrix}1\\0\end{pmatrix}\} \quad\text{and}\quad T = \{\begin{pmatrix}2\\0\end{pmatrix}\}$$

since the linear dependence of $S_1 \cup S_2$ is easily seen.

(d) The union of two linearly independent sets $S \cup T$ is linearly independent if and only if their spans have a trivial intersection $[S] \cap [T] = \{\vec{0}\}$. To prove that, assume that S and T are linearly independent subsets of some vector space.

For the 'only if' direction, assume that the intersection of the spans is trivial $[S] \cap [T] = \{\vec{0}\}$. Consider the set $S \cup T$. Any linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ gives $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = -d_1\vec{t}_1 - \cdots - d_m\vec{t}_m$. The left side of that equation sums to a vector in $[S]$, and the right side is a vector in $[T]$. Therefore, since the intersection of the spans is trivial, both sides equal the zero vector. Because S is linearly independent, all of the c's are zero. Because T is linearly independent, all of the d's are zero. Thus, the original linear relationship among members of $S \cup T$ only holds if all of the coefficients are zero. That shows that $S \cup T$ is linearly independent.

For the 'if' half we can make the same argument in reverse. If the union $S \cup T$ is linearly independent, that is, if the only solution to $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ is the trivial solution $c_1 = 0, \ldots, d_m = 0$, then any vector $\vec{v}$ in the intersection of the spans $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = -d_1\vec{t}_1 - \cdots = d_m\vec{t}_m$ must be the zero vector because each scalar is zero.

Two.II.1.37 **(a)** We do induction on the number of vectors in the finite set S.

The base case is that S has no elements. In this case S is linearly independent and there is nothing to check — a subset of S that has the same span as S is S itself.

For the inductive step assume that the theorem is true for all sets of size $n = 0$, $n = 1$, ..., $n = k$ in order to prove that it holds when S has $n = k+1$ elements. If the $k+1$-element set $S = \{\vec{s}_0, \ldots, \vec{s}_k\}$ is linearly independent then the theorem is trivial, so assume that it is dependent. By Corollary 1.17 there is an $\vec{s}_i$ that is a linear combination of other vectors in S. Define $S_1 = S - \{\vec{s}_i\}$ and note that S_1 has the same span as S by Lemma 1.1. The set S_1 has k elements and so the inductive hypothesis applies to give that it has a linearly independent subset with the same span. That subset of S_1 is the desired subset of S.

(b) Here is a sketch of the argument. The induction argument details have been left out.

If the finite set S is empty then there is nothing to prove. If $S = \{\vec{0}\}$ then the empty subset will do.

Otherwise, take some nonzero vector $\vec{s}_1 \in S$ and define $S_1 = \{\vec{s}_1\}$. If $[S_1] = [S]$ then this proof is finished by noting that S_1 is linearly independent.

If not, then there is a nonzero vector $\vec{s}_2 \in S - [S_1]$ (if every $\vec{s} \in S$ is in $[S_1]$ then $[S_1] = [S]$). Define $S_2 = S_1 \cup \{\vec{s}_2\}$. If $[S_2] = [S]$ then this proof is finished by using Theorem 1.17 to show that S_2 is linearly independent.

Repeat the last paragraph until a set with a big enough span appears. That must eventually happen because S is finite, and $[S]$ will be reached at worst when every vector from S has been used.

Two.II.1.38 **(a)** Assuming first that $a \neq 0$,

$$x\begin{pmatrix}a\\c\end{pmatrix} + y\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

gives

$$\begin{array}{rl} ax + by = 0 \\ cx + dy = 0 \end{array} \xrightarrow{-(c/a)\rho_1+\rho_2} \begin{array}{rr} ax + & by = 0 \\ & (-(c/a)b + d)y = 0 \end{array}$$

which has a solution if and only if $0 \neq -(c/a)b + d = (-cb + ad)/d$ (we've assumed in this case that $a \neq 0$, and so back substitution yields a unique solution).

The $a = 0$ case is also not hard — break it into the $c \neq 0$ and $c = 0$ subcases and note that in these cases $ad - bc = 0 \cdot d - bc$.

Comment. An earlier exercise showed that a two-vector set is linearly dependent if and only if either vector is a scalar multiple of the other. That can also be used to make the calculation.

(b) The equation

$$c_1\begin{pmatrix}a\\d\\g\end{pmatrix}+c_2\begin{pmatrix}b\\e\\h\end{pmatrix}+c_3\begin{pmatrix}c\\f\\i\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

gives rise to a homogeneous linear system. We proceed by writing it in matrix form and applying Gauss' method.

We first reduce the matrix to upper-triangular. Assume that $a\neq 0$.

$$\xrightarrow{(1/a)\rho_1}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\d&e&f&0\\g&h&i&0\end{array}\right)\xrightarrow[-g\rho_1+\rho_3]{-d\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ae-bd)/a&(af-cd)/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right)$$

$$\xrightarrow{(a/(ae-bd))\rho_2}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&1&(af-cd)/(ae-bd)&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right)$$

(where we've assumed for the moment that $ae-bd\neq 0$ in order to do the row reduction step). Then, under the assumptions, we get this.

$$\xrightarrow{((ah-bg)/a)\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&\frac{b}{a}&\frac{c}{a}&0\\0&1&\frac{af-cd}{ae-bd}&0\\0&0&\frac{aei+bgf+cdh-hfa-idb-gec}{ae-bd}&0\end{array}\right)$$

shows that the original system is nonsingular if and only if the $3,3$ entry is nonzero. This fraction is defined because of the $ae-bd\neq 0$ assumption, and it will equal zero if and only if its numerator equals zero.

We next worry about the assumptions. First, if $a\neq 0$ but $ae-bd=0$ then we swap

$$\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&0&(af-cd)/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right)\xrightarrow{\rho_2\leftrightarrow\rho_3}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\\0&0&(af-cd)/a&0\end{array}\right)$$

and conclude that the system is nonsingular if and only if either $ah-bg=0$ or $af-cd=0$. That's the same as asking that their product be zero:

$$\begin{aligned}ahaf-ahcd-bgaf+bgcd&=0\\ahaf-ahcd-bgaf+aegc&=0\\a(haf-hcd-bgf+egc)&=0\end{aligned}$$

(in going from the first line to the second we've applied the case assumption that $ae-bd=0$ by substituting ae for bd). Since we are assuming that $a\neq 0$, we have that $haf-hcd-bgf+egc=0$. With $ae-bd=0$ we can rewrite this to fit the form we need: in this $a\neq 0$ and $ae-bd=0$ case, the given system is nonsingular when $haf-hcd-bgf+egc-i(ae-bd)=0$, as required.

The remaining cases have the same character. Do the $a=0$ but $d\neq 0$ case and the $a=0$ and $d=0$ but $g\neq 0$ case by first swapping rows and then going on as above. The $a=0$, $d=0$, and $g=0$ case is easy — a set with a zero vector is linearly dependent, and the formula comes out to equal zero.

(c) It is linearly dependent if and only if either vector is a multiple of the other. That is, it is not independent iff

$$\begin{pmatrix}a\\d\\g\end{pmatrix}=r\cdot\begin{pmatrix}b\\e\\h\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}b\\e\\h\end{pmatrix}=s\cdot\begin{pmatrix}a\\d\\g\end{pmatrix}$$

(or both) for some scalars r and s. Eliminating r and s in order to restate this condition only in terms of the given letters a, b, d, e, g, h, we have that it is not independent — it is dependent — iff $ae-bd=ah-gb=dh-ge$.

(d) Dependence or independence is a function of the indices, so there is indeed a formula (although at first glance a person might think the formula involves cases: "if the first component of the first vector is zero then …", this guess turns out not to be correct).

Two.II.1.39 Recall that two vectors from $\mathbb{R}^n$ are perpendicular if and only if their dot product is zero.

(a) Assume that $\vec{v}$ and $\vec{w}$ are perpendicular nonzero vectors in $\mathbb{R}^n$, with $n>1$. With the linear relationship $c\vec{v}+d\vec{w}=\vec{0}$, apply $\vec{v}$ to both sides to conclude that $c\cdot\|\vec{v}\|^2+d\cdot 0=0$. Because $\vec{v}\neq\vec{0}$ we have that $c=0$. A similar application of $\vec{w}$ shows that $d=0$.

(b) Two vectors in $\mathbb{R}^1$ are perpendicular if and only if at least one of them is zero.

We define $\mathbb{R}^0$ to be a trivial space, and so both $\vec{v}$ and $\vec{w}$ are the zero vector.

(c) The right generalization is to look at a set $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq \mathbb{R}^k$ of vectors that are *mutually orthogonal* (also called *pairwise perpendicular*): if $i \neq j$ then $\vec{v}_i$ is perpendicular to $\vec{v}_j$. Mimicing the proof of the first item above shows that such a set of nonzero vectors is linearly independent.

Two.II.1.40 **(a)** This check is routine.

(b) The summation is infinite (has infinitely many summands). The definition of linear combination involves only finite sums.

(c) No nontrivial finite sum of members of $\{g, f_0, f_1, \ldots\}$ adds to the zero object: assume that

$$c_0 \cdot (1/(1-x)) + c_1 \cdot 1 + \cdots + c_n \cdot x^n = 0$$

(any finite sum uses a highest power, here n). Multiply both sides by $1 - x$ to conclude that each coefficient is zero, because a polynomial describes the zero function only when it is the zero polynomial.

Two.II.1.41 It is both 'if' and 'only if'.

Let T be a subset of the subspace S of the vector space V. The assertion that any linear relationship $c_1\vec{t}_1 + \cdots + c_n\vec{t}_n = \vec{0}$ among members of T must be the trivial relationship $c_1 = 0, \ldots, c_n = 0$ is a statement that holds in S if and only if it holds in V, because the subspace S inherits its addition and scalar multiplication operations from V.

Subsection Two.III.1: Basis

Two.III.1.16 By Theorem 1.12, each is a basis if and only if each vector in the space can be given in a unique way as a linear combination of the given vectors.

(a) Yes this is a basis. The relation

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & x \\ 2 & 2 & 0 & y \\ 3 & 1 & 1 & z \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 3 & 0 & x \\ 0 & -4 & 0 & -2x+y \\ 0 & 0 & 1 & x-2y+z \end{array}\right)$$

which has the unique solution $c_3 = x - 2y + z$, $c_2 = x/2 - y/4$, and $c_1 = -x/2 + 3y/4$.

(b) This is not a basis. Setting it up as in the prior item

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives a linear system whose solution

$$\left(\begin{array}{cc|c} 1 & 3 & x \\ 2 & 2 & y \\ 3 & 1 & z \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{cc|c} 1 & 3 & x \\ 0 & -4 & -2x+y \\ 0 & 0 & x-2y+z \end{array}\right)$$

is possible if and only if the three-tall vector's components x, y, and z satisfy $x - 2y + z = 0$. For instance, we can find the coefficients c_1 and c_2 that work when $x = 1$, $y = 1$, and $z = 1$. However, there are no c's that work for $x = 1$, $y = 1$, and $z = 2$. Thus this is not a basis; it does not span the space.

(c) Yes, this is a basis. Setting up the relationship leads to this reduction

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & x \\ 2 & 1 & 5 & y \\ -1 & 1 & 0 & z \end{array}\right) \xrightarrow{\rho_1\leftrightarrow\rho_3} \xrightarrow{2\rho_1+\rho_2} \xrightarrow{-(1/3)\rho_2+\rho_3} \left(\begin{array}{ccc|c} -1 & 1 & 0 & z \\ 0 & 3 & 5 & y+2z \\ 0 & 0 & 1/3 & x-y/3-2z/3 \end{array}\right)$$

which has a unique solution for each triple of components x, y, and z.

(d) No, this is not a basis. The reduction

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & x \\ 2 & 1 & 3 & y \\ -1 & 1 & 0 & z \end{array}\right) \xrightarrow{\rho_1\leftrightarrow\rho_3} \xrightarrow{2\rho_1+\rho_2} \xrightarrow{(-1/3)\rho_2+\rho_3} \left(\begin{array}{ccc|c} -1 & 1 & 0 & z \\ 0 & 3 & 3 & y+2z \\ 0 & 0 & 0 & x-y/3-2z/3 \end{array}\right)$$

which does not have a solution for each triple x, y, and z. Instead, the span of the given set includes only those three-tall vectors where $x = y/3 + 2z/3$.

Two.III.1.17 **(a)** We solve

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

with

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \end{array}\right)$$

and conclude that $c_2 = 1/2$ and so $c_1 = 3/2$. Thus, the representation is this.

$$\mathrm{Rep}_B(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}_B$$

(b) The relationship $c_1 \cdot (1) + c_2 \cdot (1 + x) + c_3 \cdot (1 + x + x^2) + c_4 \cdot (1 + x + x^2 + x^3) = x^2 + x^3$ is easily solved by eye to give that $c_4 = 1$, $c_3 = 0$, $c_2 = -1$, and $c_1 = 0$.

$$\mathrm{Rep}_D(x^2 + x^3) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}_D$$

(c) $\mathrm{Rep}_{\mathcal{E}_4}(\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{E}_4}$

Two.III.1.18 A natural basis is $\langle 1, x, x^2\rangle$. There are bases for $\mathcal{P}_2$ that do not contain any polynomials of degree one or degree zero. One is $\langle 1 + x + x^2, x + x^2, x^2\rangle$. (Every basis has at least one polynomial of degree two, though.)

Two.III.1.19 The reduction

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

gives that the only condition is that $x_1 = 4x_2 - 3x_3 + x_4$. The solution set is

$$\{\begin{pmatrix} 4x_2 - 3x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \bigm| x_2, x_3, x_4 \in \mathbb{R}\} = \{x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \bigm| x_2, x_3, x_4 \in \mathbb{R}\}$$

and so the obvious candidate for the basis is this.

$$\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

We've shown that this spans the space, and showing it is also linearly independent is routine.

Two.III.1.20 There are many bases. This is a natural one.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

Two.III.1.21 For each item, many answers are possible.

(a) One way to proceed is to parametrize by expressing the a_2 as a combination of the other two $a_2 = 2a_1 + a_0$. Then $a_2x^2 + a_1x + a_0$ is $(2a_1 + a_0)x^2 + a_1x + a_0$ and

$$\{(2a_1 + a_0)x^2 + a_1x + a_0 \bigm| a_1, a_0 \in \mathbb{R}\} = \{a_1 \cdot (2x^2 + x) + a_0 \cdot (x^2 + 1) \bigm| a_1, a_0 \in \mathbb{R}\}$$

suggests $\langle 2x^2 + x, x^2 + 1\rangle$. This only shows that it spans, but checking that it is linearly independent is routine.

(b) Parametrize $\{\begin{pmatrix} a & b & c \end{pmatrix} \bigm| a + b = 0\}$ to get $\{\begin{pmatrix} -b & b & c \end{pmatrix} \bigm| b, c \in \mathbb{R}\}$, which suggests using the sequence $\langle \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \rangle$. We've shown that it spans, and checking that it is linearly independent is easy.

(c) Rewriting

$$\{\begin{pmatrix} a & b \\ 0 & 2b \end{pmatrix} \bigm| a, b \in \mathbb{R}\} = \{a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \bigm| a, b \in \mathbb{R}\}$$

suggests this for the basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \rangle$$

Two.III.1.22 We will show that the second is a basis; the first is similar. We will show this straight from the definition of a basis, because this example appears before Theorem 1.12.

To see that it is linearly independent, we set up $c_1 \cdot (\cos\theta - \sin\theta) + c_2 \cdot (2\cos\theta + 3\sin\theta) = 0\cos\theta + 0\sin\theta$. Taking $\theta = 0$ and $\theta = \pi/2$ gives this system

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 2 &= 0 \\ c_1 \cdot (-1) + c_2 \cdot 3 &= 0 \end{aligned} \xrightarrow{\rho_1+\rho_2} \begin{aligned} c_1 + 2c_2 &= 0 \\ +5c_2 &= 0 \end{aligned}$$

which shows that $c_1 = 0$ and $c_2 = 0$.

The calculation for span is also easy; for any $x, y \in \mathbb{R}$, we have that $c_1 \cdot (\cos\theta - \sin\theta) + c_2 \cdot (2\cos\theta + 3\sin\theta) = x\cos\theta + y\sin\theta$ gives that $c_2 = x/5 + y/5$ and that $c_1 = 3x/5 - 2y/5$, and so the span is the entire space.

Two.III.1.23 **(a)** Asking which $a_0 + a_1x + a_2x^2$ can be expressed as $c_1 \cdot (1 + x) + c_2 \cdot (1 + 2x)$ gives rise to three linear equations, describing the coefficients of x^2, x, and the constants.

$$\begin{aligned} c_1 + c_2 &= a_0 \\ c_1 + 2c_2 &= a_1 \\ 0 &= a_2 \end{aligned}$$

Gauss' method with back-substitution shows, provided that $a_2 = 0$, that $c_2 = -a_0 + a_1$ and $c_1 = 2a_0 - a_1$. Thus, with $a_2 = 0$, we can compute appropriate c_1 and c_2 for any a_0 and a_1. So the span is the entire set of linear polynomials $\{a_0 + a_1x \bigm| a_0, a_1 \in \mathbb{R}\}$. Parametrizing that set $\{a_0 \cdot 1 + a_1 \cdot x \bigm| a_0, a_1 \in \mathbb{R}\}$ suggests a basis $\langle 1, x\rangle$ (we've shown that it spans; checking linear independence is easy).

(b) With

$$a_0 + a_1x + a_2x^2 = c_1 \cdot (2 - 2x) + c_2 \cdot (3 + 4x^2) = (2c_1 + 3c_2) + (-2c_1)x + (4c_2)x^2$$

we get this system.

$$\begin{aligned} 2c_1 + 3c_2 &= a_0 \\ -2c_1 \qquad &= a_1 \\ 4c_2 &= a_2 \end{aligned} \xrightarrow{\rho_1+\rho_2} \xrightarrow{(-4/3)\rho_2+\rho_3} \begin{aligned} 2c_1 + 3c_2 &= a_0 \\ 3c_2 &= a_0 + a_1 \\ 0 &= (-4/3)a_0 - (4/3)a_1 + a_2 \end{aligned}$$

Thus, the only quadratic polynomials $a_0 + a_1x + a_2x^2$ with associated c's are the ones such that $0 = (-4/3)a_0 - (4/3)a_1 + a_2$. Hence the span is $\{(-a_1 + (3/4)a_2) + a_1x + a_2x^2 \bigm| a_1, a_2 \in \mathbb{R}\}$. Parametrizing gives $\{a_1 \cdot (-1 + x) + a_2 \cdot ((3/4) + x^2) \bigm| a_1, a_2 \in \mathbb{R}\}$, which suggests $\langle -1+x, (3/4)+x^2\rangle$ (checking that it is linearly independent is routine).

Two.III.1.24 **(a)** The subspace is $\{a_0 + a_1x + a_2x^2 + a_3x^3 \bigm| a_0 + 7a_1 + 49a_2 + 343a_3 = 0\}$. Rewriting $a_0 = -7a_1 - 49a_2 - 343a_3$ gives $\{(-7a_1 - 49a_2 - 343a_3) + a_1x + a_2x^2 + a_3x^3 \bigm| a_1, a_2, a_3 \in \mathbb{R}\}$, which, on breaking out the parameters, suggests $\langle -7 + x, -49 + x^2, -343 + x^3\rangle$ for the basis (it is easily verified).

(b) The given subspace is the collection of cubics $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$ and $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$. Gauss' method

$$\begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 0 \end{aligned} \xrightarrow{-\rho_1+\rho_2} \begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ -2a_1 - 24a_2 - 218a_3 &= 0 \end{aligned}$$

gives that $a_1 = -12a_2 - 109a_3$ and that $a_0 = 35a_2 + 420a_3$. Rewriting $(35a_2 + 420a_3) + (-12a_2 - 109a_3)x + a_2x^2 + a_3x^3$ as $a_2 \cdot (35 - 12x + x^2) + a_3 \cdot (420 - 109x + x^3)$ suggests this for a basis $\langle 35 - 12x + x^2, 420 - 109x + x^3\rangle$. The above shows that it spans the space. Checking it is linearly independent is routine. (*Comment.* A worthwhile check is to verify that both polynomials in the basis have both seven and five as roots.)

(c) Here there are three conditions on the cubics, that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$, that $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$, and that $a_0 + 3a_1 + 9a_2 + 27a_3 = 0$. Gauss' method

$$\begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 0 \\ a_0 + 3a_1 + 9a_2 + 27a_3 &= 0 \end{aligned} \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \begin{aligned} a_0 + 7a_1 + 49a_2 + 343a_3 &= 0 \\ -2a_1 - 24a_2 - 218a_3 &= 0 \\ 8a_2 + 120a_3 &= 0 \end{aligned}$$

yields the single free variable a_3, with $a_2 = -15a_3$, $a_1 = 71a_3$, and $a_0 = -105a_3$. The parametrization is this.

$$\{(-105a_3) + (71a_3)x + (-15a_3)x^2 + (a_3)x^3 \bigm| a_3 \in \mathbb{R}\} = \{a_3 \cdot (-105 + 71x - 15x^2 + x^3) \bigm| a_3 \in \mathbb{R}\}$$

Therefore, a natural candidate for the basis is $\langle -105 + 71x - 15x^2 + x^3\rangle$. It spans the space by the work above. It is clearly linearly independent because it is a one-element set (with that single element not the zero object of the space). Thus, any cubic through the three points $(7,0)$, $(5,0)$, and $(3,0)$ is a multiple of this one. (*Comment.* As in the prior question, a worthwhile check is to verify that plugging seven, five, and three into this polynomial yields zero each time.)

(d) This is the trivial subspace of $\mathcal{P}_3$. Thus, the basis is empty $\langle\rangle$.

Remark. The polynomial in the third item could alternatively have been derived by multiplying out $(x-7)(x-5)(x-3)$.

Two.III.1.25 Yes. Linear independence and span are unchanged by reordering.

Two.III.1.26 No linearly independent set contains a zero vector.

Two.III.1.27 **(a)** To show that it is linearly independent, note that $d_1(c_1\vec{\beta}_1)+d_2(c_2\vec{\beta}_2)+d_3(c_3\vec{\beta}_3) = \vec{0}$ gives that $(d_1c_1)\vec{\beta}_1 + (d_2c_2)\vec{\beta}_2 + (d_3c_3)\vec{\beta}_3 = \vec{0}$, which in turn implies that each d_ic_i is zero. But with $c_i \neq 0$ that means that each d_i is zero. Showing that it spans the space is much the same; because $\langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ is a basis, and so spans the space, we can for any $\vec{v}$ write $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$, and then $\vec{v} = (d_1/c_1)(c_1\vec{\beta}_1) + (d_2/c_2)(c_2\vec{\beta}_2) + (d_3/c_3)(c_3\vec{\beta}_3)$.

If any of the scalars are zero then the result is not a basis, because it is not linearly independent.

(b) Showing that $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ is linearly independent is easy. To show that it spans the space, assume that $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$. Then, we can represent the same $\vec{v}$ with respect to $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ in this way $\vec{v} = (1/2)(d_1 - d_2 - d_3)(2\vec{\beta}_1) + d_2(\vec{\beta}_1 + \vec{\beta}_2) + d_3(\vec{\beta}_1 + \vec{\beta}_3)$.

Two.III.1.28 Each forms a linearly independent set if $\vec{v}$ is ommitted. To preserve linear independence, we must expand the span of each. That is, we must determine the span of each (leaving $\vec{v}$ out), and then pick a $\vec{v}$ lying outside of that span. Then to finish, we must check that the result spans the entire given space. Those checks are routine.

(a) Any vector that is not a multiple of the given one, that is, any vector that is not on the line $y = x$ will do here. One is $\vec{v} = \vec{e}_1$.

(b) By inspection, we notice that the vector $\vec{e}_3$ is not in the span of the set of the two given vectors. The check that the resulting set is a basis for $\mathbb{R}^3$ is routine.

(c) For any member of the span $\{c_1 \cdot (x) + c_2 \cdot (1 + x^2) \bigm| c_1, c_2 \in \mathbb{R}\}$, the coefficient of x^2 equals the constant term. So we expand the span if we add a quadratic without this property, say, $\vec{v} = 1 - x^2$. The check that the result is a basis for $\mathcal{P}_2$ is easy.

Two.III.1.29 To show that each scalar is zero, simply subtract $c_1\vec{\beta}_1+\cdots+c_k\vec{\beta}_k-c_{k+1}\vec{\beta}_{k+1}-\cdots-c_n\vec{\beta}_n = \vec{0}$. The obvious generalization is that in any equation involving only the $\vec{\beta}$'s, and in which each $\vec{\beta}$ appears only once, each scalar is zero. For instance, an equation with a combination of the even-indexed basis vectors (i.e., $\vec{\beta}_2$, $\vec{\beta}_4$, etc.) on the right and the odd-indexed basis vectors on the left also gives the conclusion that all of the coefficients are zero.

Two.III.1.30 No; no linearly independent set contains the zero vector.

Two.III.1.31 Here is a subset of $\mathbb{R}^2$ that is not a basis, and two different linear combinations of its elements that sum to the same vector.

$$\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\} \qquad 2\cdot\begin{pmatrix}1\\2\end{pmatrix} + 0\cdot\begin{pmatrix}2\\4\end{pmatrix} = 0\cdot\begin{pmatrix}1\\2\end{pmatrix} + 1\cdot\begin{pmatrix}2\\4\end{pmatrix}$$

Thus, when a subset is not a basis, it can be the case that its linear combinations are not unique.

But just because a subset is not a basis does not imply that its combinations must be not unique. For instance, this set

$$\{\begin{pmatrix}1\\2\end{pmatrix}\}$$

does have the property that

$$c_1\cdot\begin{pmatrix}1\\2\end{pmatrix} = c_2\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

implies that $c_1 = c_2$. The idea here is that this subset fails to be a basis because it fails to span the space; the proof of the theorem establishes that linear combinations are unique if and only if the subset is linearly independent.

Two.III.1.32 **(a)** Describing the vector space as

$$\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

suggests this for a basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$$

Verification is easy.

(b) This is one possible basis.

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rangle$$

(c) As in the prior two questions, we can form a basis from two kinds of matrices. First are the matrices with a single one on the diagonal and all other entries zero (there are n of those matrices). Second are the matrices with two opposed off-diagonal entries are ones and all other entries are zeros. (That is, all entries in M are zero except that $m_{i,j}$ and $m_{j,i}$ are one.)

Two.III.1.33 **(a)** Any four vectors from $\mathbb{R}^3$ are linearly related because the vector equation

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + c_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} + c_4 \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to a linear system

$$\begin{aligned} x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 &= 0 \\ y_1c_1 + y_2c_2 + y_3c_3 + y_4c_4 &= 0 \\ z_1c_1 + z_2c_2 + z_3c_3 + z_4c_4 &= 0 \end{aligned}$$

that is homogeneous (and so has a solution) and has four unknowns but only three equations, and therefore has nontrivial solutions. (Of course, this argument applies to any subset of $\mathbb{R}^3$ with four or more vectors.)

(b) We shall do just the two-vector case. Given $x_1, \ldots, z_2$,

$$S = \{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\}$$

to decide which vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

are in the span of S, set up

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and row reduce the resulting system.

$$\begin{aligned} x_1c_1 + x_2c_2 &= x \\ y_1c_1 + y_2c_2 &= y \\ z_1c_1 + z_2c_2 &= z \end{aligned}$$

There are two variables c_1 and c_2 but three equations, so when Gauss' method finishes, on the bottom row there will be some relationship of the form $0 = m_1x + m_2y + m_3z$. Hence, vectors in the span of the two-element set S must satisfy some restriction. Hence the span is not all of $\mathbb{R}^3$.

Two.III.1.34 We have (using these peculiar operations with care)

$$\{\begin{pmatrix} 1-y-z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R}\} = \{\begin{pmatrix} -y+1 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z+1 \\ 0 \\ z \end{pmatrix} \mid y, z \in \mathbb{R}\} = \{y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R}\}$$

and so a natural candidate for a basis is this.

$$\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

To check linear independence we set up

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(the vector on the right is the zero object in this space). That yields the linear system

$$\begin{array}{rl} (-c_1 + 1) + (-c_2 + 1) - 1 &= 1 \\ c_1 \qquad\qquad &= 0 \\ c_2 &= 0 \end{array}$$

with only the solution $c_1 = 0$ and $c_2 = 0$. Checking the span is similar.

Subsection Two.III.2: Dimension

Two.III.2.14 One basis is $\langle 1, x, x^2 \rangle$, and so the dimension is three.

Two.III.2.15 The solution set is

$$\{ \begin{pmatrix} 4x_2 - 3x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid x_2, x_3, x_4 \in \mathbb{R} \}$$

so a natural basis is this

$$\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

(checking linear independence is easy). Thus the dimension is three.

Two.III.2.16 For this space

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \} = \{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \cdots + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \}$$

this is a natural basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The dimension is four.

Two.III.2.17 **(a)** As in the prior exercise, the space $\mathcal{M}_{2\times2}$ of matrices without restriction has this basis

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

and so the dimension is four.

(b) For this space

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = b - 2c \text{ and } d \in \mathbb{R} \} = \{ b \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid b, c, d \in \mathbb{R} \}$$

this is a natural basis.

$$\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The dimension is three.

(c) Gauss' method applied to the two-equation linear system gives that $c = 0$ and that $a = -b$. Thus, we have this description

$$\{ \begin{pmatrix} -b & b \\ 0 & d \end{pmatrix} \mid b, d \in \mathbb{R} \} = \{ b \cdot \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid b, d \in \mathbb{R} \}$$

and so this is a natural basis.

$$\langle \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The dimension is two.

Two.III.2.18 The bases for these spaces are developed in the answer set of the prior subsection.

(a) One basis is $\langle -7+x, -49+x^2, -343+x^3\rangle$. The dimension is three.
(b) One basis is $\langle 35-12x+x^2, 420-109x+x^3\rangle$ so the dimension is two.
(c) A basis is $\{-105+71x-15x^2+x^3\}$. The dimension is one.
(d) This is the trivial subspace of $\mathcal{P}_3$ and so the basis is empty. The dimension is zero.

Two.III.2.19 First recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and so deletion of $\cos 2\theta$ from this set leaves the span unchanged. What's left, the set $\{\cos^2\theta, \sin^2\theta, \sin 2\theta\}$, is linearly independent (consider the relationship $c_1\cos^2\theta + c_2\sin^2\theta + c_3\sin 2\theta = Z(\theta)$ where Z is the zero function, and then take $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$ to conclude that each c is zero). It is therefore a basis for its span. That shows that the span is a dimension three vector space.

Two.III.2.20 Here is a basis

$$\langle (1+0i, 0+0i, \ldots, 0+0i),\ (0+1i, 0+0i, \ldots, 0+0i), (0+0i, 1+0i, \ldots, 0+0i), \ldots\rangle$$

and so the dimension is $2\cdot 47 = 94$.

Two.III.2.21 A basis is

$$\langle \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \ldots, \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{pmatrix}\rangle$$

and thus the dimension is $3\cdot 5 = 15$.

Two.III.2.22 In a four-dimensional space a set of four vectors is linearly independent if and only if it spans the space. The form of these vectors makes linear independence easy to show (look at the equation of fourth components, then at the equation of third components, etc.).

Two.III.2.23 **(a)** The diagram for $\mathcal{P}_2$ has four levels. The top level has the only three-dimensional subspace, $\mathcal{P}_2$ itself. The next level contains the two-dimensional subspaces (*not* just the linear polynomials; any two-dimensional subspace, like those polynomials of the form ax^2+b). Below that are the one-dimensional subspaces. Finally, of course, is the only zero-dimensional subspace, the trivial subspace.
(b) For $\mathcal{M}_{2\times 2}$, the diagram has five levels, including subspaces of dimension four through zero.

Two.III.2.24 **(a)** One **(b)** Two **(c)** n

Two.III.2.25 We need only produce an infinite linearly independent set. One is $\langle f_1, f_2, \ldots\rangle$ where $f_i\colon \mathbb{R}\to\mathbb{R}$ is

$$f_i(x) = \begin{cases} 1 & \text{if } x = i \\ 0 & \text{otherwise} \end{cases}$$

the function that has value 1 only at $x = i$.

Two.III.2.26 A function is a set of ordered pairs $(x, f(x))$. So there is only one function with an empty domain, namely the empty set. A vector space with only one element a trivial vector space and has dimension zero.

Two.III.2.27 Apply Corollary 2.8.

Two.III.2.28 A plane has the form $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 \mid t_1, t_2 \in \mathbb{R}\}$. (The first chapter also calls this a '2-flat', and contains a discussion of why this is equivalent to the description often taken in Calculus as the set of points (x, y, z) subject to a condition of the form $ax + by + cz = d$). When the plane passes through the origin we can take the particular vector $\vec{p}$ to be $\vec{0}$. Thus, in the language we have developed in this chapter, a plane through the origin is the span of a set of two vectors.

Now for the statement. Asserting that the three are not coplanar is the same as asserting that no vector lies in the span of the other two — no vector is a linear combination of the other two. That's simply an assertion that the three-element set is linearly independent. By Corollary 2.12, that's equivalent to an assertion that the set is a basis for $\mathbb{R}^3$ (more precisely, any sequence made from the set's elements is a basis).

Two.III.2.29 Let the space V be finite dimensional. Let S be a subspace of V.
(a) The empty set is a linearly independent subset of S. By Corollary 2.10, it can be expanded to a basis for the vector space S.
(b) Any basis for the subspace S is a linearly independent set in the superspace V. Hence it can be expanded to a basis for the superspace, which is finite dimensional. Therefore it has only finitely many members.

Two.III.2.30 It ensures that we exhaust the $\vec{\beta}$'s. That is, it justifies the first sentence of the last paragraph.

Two.III.2.31 Let B_U be a basis for U and let B_W be a basis for W. The set $B_U \cup B_W$ is linearly dependent as it is a six member subset of the five-dimensional space $\mathbb{R}^5$. Thus some member of B_W is in the span of B_U, and thus $U \cap W$ is more than just the trivial space $\{\vec{0}\,\}$.

Generalization: if U, W are subspaces of a vector space of dimension n and if $\dim(U)+\dim(W) > n$ then they have a nontrivial intersection.

Two.III.2.32 First, note that a set is a basis for some space if and only if it is linearly independent, because in that case it is a basis for its own span.

(a) The answer to the question in the second paragraph is "yes" (implying "yes" answers for both questions in the first paragraph). If B_U is a basis for U then B_U is a linearly independent subset of W. Apply Corollary 2.10 to expand it to a basis for W. That is the desired B_W.

The answer to the question in the third paragraph is "no", which implies a "no" answer to the question of the fourth paragraph. Here is an example of a basis for a superspace with no sub-basis forming a basis for a subspace: in $W = \mathbb{R}^2$, consider the standard basis $\mathcal{E}_2$. No sub-basis of $\mathcal{E}_2$ forms a basis for the subspace U of $\mathbb{R}^2$ that is the line $y = x$.

(b) It is a basis (for its span) because the intersection of linearly independent sets is linearly independent (the intersection is a subset of each of the linearly independent sets).

It is not, however, a basis for the intersection of the spaces. For instance, these are bases for $\mathbb{R}^2$:

$$B_1 = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle \quad\text{and}\quad B_2 = \langle \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \rangle$$

and $\mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$, but $B_1 \cap B_2$ is empty. All we can say is that the $\cap$ of the bases is a basis for a subset of the intersection of the spaces.

(c) The $\cup$ of bases need not be a basis: in $\mathbb{R}^2$

$$B_1 = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \rangle \quad\text{and}\quad B_2 = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \rangle$$

$B_1 \cup B_2$ is not linearly independent. A necessary and sufficient condition for a $\cup$ of two bases to be a basis

$$B_1 \cup B_2 \text{ is linearly independent} \quad\Longleftrightarrow\quad [B_1 \cap B_2] = [B_1] \cap [B_2]$$

it is easy enough to prove (but perhaps hard to apply).

(d) The complement of a basis cannot be a basis because it contains the zero vector.

Two.III.2.33 **(a)** A basis for U is a linearly independent set in W and so can be expanded via Corollary 2.10 to a basis for W. The second basis has at least as many members as the first.

(b) One direction is clear: if $V = W$ then they have the same dimension. For the converse, let B_U be a basis for U. It is a linearly independent subset of W and so can be expanded to a basis for W. If $\dim(U) = \dim(W)$ then this basis for W has no more members than does B_U and so equals B_U. Since U and W have the same bases, they are equal.

(c) Let W be the space of finite-degree polynomials and let U be the subspace of polynomails that have only even-powered terms $\{a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n} \mid a_0, \ldots, a_n \in \mathbb{R}\}$. Both spaces have infinite dimension, but U is a proper subspace.

Two.III.2.34 The possibilities for the dimension of V are 0, 1, $n-1$, and n.

To see this, first consider the case when all the coordinates of $\vec{v}$ are equal.

$$\vec{v} = \begin{pmatrix}z\\z\\\vdots\\z\end{pmatrix}$$

Then $\sigma(\vec{v}) = \vec{v}$ for every permutation σ, so V is just the span of $\vec{v}$, which has dimension 0 or 1 according to whether $\vec{v}$ is $\vec{0}$ or not.

Now suppose not all the coordinates of $\vec{v}$ are equal; let x and y with $x \neq y$ be among the coordinates of $\vec{v}$. Then we can find permutations σ_1 and σ_2 such that

$$\sigma_1(\vec{v}) = \begin{pmatrix}x\\y\\a_3\\\vdots\\a_n\end{pmatrix} \quad\text{and}\quad \sigma_2(\vec{v}) = \begin{pmatrix}y\\x\\a_3\\\vdots\\a_n\end{pmatrix}$$

for some $a_3, \dots, a_n \in \mathbb{R}$. Therefore,

$$\frac{1}{y-x}\big(\sigma_1(\vec{v})-\sigma_2(\vec{v})\big)=\begin{pmatrix}-1\\1\\0\\ \vdots\\0\end{pmatrix}$$

is in V. That is, $\vec{e}_2-\vec{e}_1\in V$, where $\vec{e}_1$, $\vec{e}_2$, ..., $\vec{e}_n$ is the standard basis for $\mathbb{R}^n$. Similarly, $\vec{e}_3-\vec{e}_2$, ..., $\vec{e}_n-\vec{e}_1$ are all in V. It is easy to see that the vectors $\vec{e}_2-\vec{e}_1$, $\vec{e}_3-\vec{e}_2$, ..., $\vec{e}_n-\vec{e}_1$ are linearly independent (that is, form a linearly independent set), so $\dim V\geq n-1$.

Finally, we can write

$$\begin{aligned}\vec{v}&=x_1\vec{e}_1+x_2\vec{e}_2+\cdots+x_n\vec{e}_n\\&=(x_1+x_2+\cdots+x_n)\vec{e}_1+x_2(\vec{e}_2-\vec{e}_1)+\cdots+x_n(\vec{e}_n-\vec{e}_1)\end{aligned}$$

This shows that if $x_1+x_2+\cdots+x_n=0$ then $\vec{v}$ is in the span of $\vec{e}_2-\vec{e}_1$, ..., $\vec{e_n}-\vec{e}_1$ (that is, is in the span of the set of those vectors); similarly, each $\sigma(\vec{v})$ will be in this span, so V will equal this span and $\dim V=n-1$. On the other hand, if $x_1+x_2+\cdots+x_n\neq 0$ then the above equation shows that $\vec{e}_1\in V$ and thus $\vec{e}_1,\dots,\vec{e}_n\in V$, so $V=\mathbb{R}^n$ and $\dim V=n$.

Subsection Two.III.3: Vector Spaces and Linear Systems

Two.III.3.16 **(a)** $\begin{pmatrix}2&3\\1&1\end{pmatrix}$ **(b)** $\begin{pmatrix}2&1\\1&3\end{pmatrix}$ **(c)** $\begin{pmatrix}1&6\\4&7\\3&8\end{pmatrix}$ **(d)** $\begin{pmatrix}0&0&0\end{pmatrix}$ **(e)** $\begin{pmatrix}-1\\-2\end{pmatrix}$

Two.III.3.17 **(a)** Yes. To see if there are c_1 and c_2 such that $c_1\cdot\begin{pmatrix}2&1\end{pmatrix}+c_2\cdot\begin{pmatrix}3&1\end{pmatrix}=\begin{pmatrix}1&0\end{pmatrix}$ we solve

$$\begin{aligned}2c_1+3c_2&=1\\c_1+c_2&=0\end{aligned}$$

and get $c_1=-1$ and $c_2=1$. Thus the vector is in the row space.

(b) No. The equation $c_1\begin{pmatrix}0&1&3\end{pmatrix}+c_2\begin{pmatrix}-1&0&1\end{pmatrix}+c_3\begin{pmatrix}-1&2&7\end{pmatrix}=\begin{pmatrix}1&1&1\end{pmatrix}$ has no solution.

$$\left(\begin{array}{ccc|c}0&-1&-1&1\\1&0&2&1\\3&1&7&1\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_2}\xrightarrow{-3\rho_1+\rho_2}\xrightarrow{\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&0&2&1\\0&-1&-1&1\\0&0&0&-1\end{array}\right)$$

Thus, the vector is not in the row space.

Two.III.3.18 **(a)** No. To see if there are $c_1, c_2\in\mathbb{R}$ such that

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

we can use Gauss' method on the resulting linear system.

$$\begin{aligned}c_1+c_2&=1\\c_1+c_2&=3\end{aligned}\;\xrightarrow{-\rho_1+\rho_2}\;\begin{aligned}c_1+c_2&=1\\0&=2\end{aligned}$$

There is no solution and so the vector is not in the column space.

(b) Yes. From this relationship

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2\begin{pmatrix}3\\0\\-3\end{pmatrix}+c_3\begin{pmatrix}1\\4\\3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

we get a linear system that, when Gauss' method is applied,

$$\left(\begin{array}{ccc|c}1&3&1&1\\2&0&4&0\\1&-3&-3&0\end{array}\right)\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&3&1&1\\0&-6&2&-2\\0&0&-6&1\end{array}\right)$$

yields a solution. Thus, the vector is in the column space.

Two.III.3.19 A routine Gaussian reduction

$$\begin{pmatrix}2&0&3&4\\0&1&1&-1\\3&1&0&2\\1&0&-4&1\end{pmatrix}\xrightarrow[-(1/2)\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3}\xrightarrow{-\rho_2+\rho_3}\xrightarrow{-\rho_3+\rho_4}\begin{pmatrix}2&0&3&4\\0&1&1&-1\\0&0&-11/2&-3\\0&0&0&0\end{pmatrix}$$

suggests this basis $\langle\begin{pmatrix}2&0&3&4\end{pmatrix},\begin{pmatrix}0&1&1&-1\end{pmatrix},\begin{pmatrix}0&0&-11/2&-3\end{pmatrix}\rangle$.

Another, perhaps more convenient procedure, is to swap rows first,

$$\xrightarrow{\rho_1\leftrightarrow\rho_4}\xrightarrow[-2\rho_1+\rho_4]{-3\rho_1+\rho_3}\xrightarrow{-\rho_2+\rho_3}\xrightarrow{-\rho_3+\rho_4}\begin{pmatrix}1&0&-4&-1\\0&1&1&-1\\0&0&11&6\\0&0&0&0\end{pmatrix}$$

leading to the basis $\langle\begin{pmatrix}1&0&-4&-1\end{pmatrix},\begin{pmatrix}0&1&1&-1\end{pmatrix},\begin{pmatrix}0&0&11&6\end{pmatrix}\rangle$.

Two.III.3.20 **(a)** This reduction

$$\xrightarrow[-(1/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-(1/3)\rho_2+\rho_3}\begin{pmatrix}2&1&3\\0&-3/2&1/2\\0&0&4/3\end{pmatrix}$$

shows that the row rank, and hence the rank, is three.

(b) Inspection of the columns shows that that the others are multiples of the first (inspection of the rows shows the same thing). Thus the rank is one.

Alternatively, the reduction

$$\begin{pmatrix}1&-1&2\\3&-3&6\\-2&2&-4\end{pmatrix}\xrightarrow[2\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}1&-1&2\\0&0&0\\0&0&0\end{pmatrix}$$

shows the same thing.

(c) This calculation

$$\begin{pmatrix}1&3&2\\5&1&1\\6&4&3\end{pmatrix}\xrightarrow[-6\rho_1+\rho_3]{-5\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&3&2\\0&-14&-9\\0&0&0\end{pmatrix}$$

shows that the rank is two.

(d) The rank is zero.

Two.III.3.21 **(a)** This reduction

$$\begin{pmatrix}1&3\\-1&3\\1&4\\2&1\end{pmatrix}\xrightarrow[\substack{-\rho_1+\rho_3\\-2\rho_1+\rho_4}]{\rho_1+\rho_2}\xrightarrow[(5/6)\rho_2+\rho_4]{-(1/6)\rho_2+\rho_3}\begin{pmatrix}1&3\\0&6\\0&0\\0&0\end{pmatrix}$$

gives $\langle\begin{pmatrix}1&3\end{pmatrix},\begin{pmatrix}0&6\end{pmatrix}\rangle$.

(b) Transposing and reducing

$$\begin{pmatrix}1&2&1\\3&1&-1\\1&-3&-3\end{pmatrix}\xrightarrow[-\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}1&2&1\\0&-5&-4\\0&-5&-4\end{pmatrix}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&2&1\\0&-5&-4\\0&0&0\end{pmatrix}$$

and then transposing back gives this basis.

$$\langle\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\-5\\-4\end{pmatrix}\rangle$$

(c) Notice first that the surrounding space is given as $\mathcal{P}_3$, not $\mathcal{P}_2$. Then, taking the first polynomial $1+1\cdot x+0\cdot x^2+0\cdot x^3$ to be "the same" as the row vector $\begin{pmatrix}1&1&0&0\end{pmatrix}$, etc., leads to

$$\begin{pmatrix}1&1&0&0\\1&0&-1&0\\3&2&-1&0\end{pmatrix}\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&1&0&0\\0&-1&-1&0\\0&0&0&0\end{pmatrix}$$

which yields the basis $\langle 1+x,-x-x^2\rangle$.

(d) Here "the same" gives

$$\begin{pmatrix}1&0&1&3&1&-1\\1&0&3&2&1&4\\-1&0&-5&-1&-1&-9\end{pmatrix}\xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{2\rho_2+\rho_3}\begin{pmatrix}1&0&1&3&1&-1\\0&0&2&-1&0&5\\0&0&0&0&0&0\end{pmatrix}$$

leading to this basis.

$$\langle\begin{pmatrix}1&0&1\\3&1&-1\end{pmatrix},\begin{pmatrix}0&0&2\\-1&0&5\end{pmatrix}\rangle$$

Two.III.3.22 Only the zero matrices have rank of zero. The only matrices of rank one have the form

$$\begin{pmatrix}k_1\cdot\rho\\ \vdots\\ k_m\cdot\rho\end{pmatrix}$$

where ρ is some nonzero row vector, and not all of the k_i's are zero. (*Remark.* We can't simply say that all of the rows are multiples of the first because the first row might be the zero row. *Another Remark.* The above also applies with 'column' replacing 'row'.)

Two.III.3.23 If $a\neq 0$ then a choice of $d=(c/a)b$ will make the second row be a multiple of the first, specifically, c/a times the first. If $a=0$ and $b=0$ then any non-0 choice for d will ensure that the second row is nonzero. If $a=0$ and $b\neq 0$ and $c=0$ then any choice for d will do, since the matrix will automatically have rank one (even with the choice of $d=0$). Finally, if $a=0$ and $b\neq 0$ and $c\neq 0$ then no choice for d will suffice because the matrix is sure to have rank two.

Two.III.3.24 The column rank is two. One way to see this is by inspection — the column space consists of two-tall columns and so can have a dimension of at least two, and we can easily find two columns that together form a linearly independent set (the fourth and fifth columns, for instance). Another way to see this is to recall that the column rank equals the row rank, and to perform Gauss' method, which leaves two nonzero rows.

Two.III.3.25 We apply Theorem 3.13. The number of columns of a matrix of coefficients A of a linear system equals the number n of unknowns. A linear system with at least one solution has at most one solution if and only if the space of solutions of the associated homogeneous system has dimension zero (recall: in the 'General = Particular + Homogeneous' equation $\vec{v}=\vec{p}+\vec{h}$, provided that such a $\vec{p}$ exists, the solution $\vec{v}$ is unique if and only if the vector $\vec{h}$ is unique, namely $\vec{h}=\vec{0}$). But that means, by the theorem, that $n=r$.

Two.III.3.26 The set of columns must be dependent because the rank of the matrix is at most five while there are nine columns.

Two.III.3.27 There is little danger of their being equal since the row space is a set of row vectors while the column space is a set of columns (unless the matrix is 1×1, in which case the two spaces must be equal).

Remark. Consider

$$A=\begin{pmatrix}1&3\\2&6\end{pmatrix}$$

and note that the row space is the set of all multiples of $\begin{pmatrix}1&3\end{pmatrix}$ while the column space consists of multiples of

$$\begin{pmatrix}1\\2\end{pmatrix}$$

so we also cannot argue that the two spaces must be simply transposes of each other.

Two.III.3.28 First, the vector space is the set of four-tuples of real numbers, under the natural operations. Although this is not the set of four-wide row vectors, the difference is slight — it is "the same" as that set. So we will treat the four-tuples like four-wide vectors.

With that, one way to see that $(1,0,1,0)$ is not in the span of the first set is to note that this reduction

$$\begin{pmatrix}1&-1&2&-3\\1&1&2&0\\3&-1&6&-6\end{pmatrix}\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&-1&2&-3\\0&2&0&3\\0&0&0&0\end{pmatrix}$$

and this one

$$\begin{pmatrix}1&-1&2&-3\\1&1&2&0\\3&-1&6&-6\\1&0&1&0\end{pmatrix}\xrightarrow[\substack{-3\rho_1+\rho_3\\-\rho_1+\rho_4}]{-\rho_1+\rho_2}\xrightarrow[-(1/2)\rho_2+\rho_4]{-\rho_2+\rho_3}\xrightarrow{\rho_3\leftrightarrow\rho_4}\begin{pmatrix}1&-1&2&-3\\0&2&0&3\\0&0&-1&3/2\\0&0&0&0\end{pmatrix}$$

yield matrices differing in rank. This means that addition of $(1,0,1,0)$ to the set of the first three four-tuples increases the rank, and hence the span, of that set. Therefore $(1,0,1,0)$ is not already in the span.

Two.III.3.29 It is a subspace because it is the column space of the matrix

$$\begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

of coefficients. To find a basis for the column space,

$$\{c_1\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} \bigm| c_1, c_2, c_3 \in \mathbb{R}\}$$

we take the three vectors from the spanning set, transpose, reduce,

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \\ 4 & -1 & 5 \end{pmatrix} \xrightarrow[-(4/3)\rho_1+\rho_3]{-(2/3)\rho_1+\rho_2} \xrightarrow{-(7/2)\rho_2+\rho_3} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -2/3 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

and transpose back to get this.

$$\langle \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2/3 \\ 2/3 \end{pmatrix} \rangle$$

Two.III.3.30 This can be done as a straightforward calculation.

$$\begin{aligned}
(rA+sB)^{\text{trans}} &= \begin{pmatrix} ra_{1,1}+sb_{1,1} & \ldots & ra_{1,n}+sb_{1,n} \\ \vdots & & \vdots \\ ra_{m,1}+sb_{m,1} & \ldots & ra_{m,n}+sb_{m,n} \end{pmatrix}^{\text{trans}} \\
&= \begin{pmatrix} ra_{1,1}+sb_{1,1} & \ldots & ra_{m,1}+sb_{m,1} \\ \vdots & & \\ ra_{1,n}+sb_{1,n} & \ldots & ra_{m,n}+sb_{m,n} \end{pmatrix} \\
&= \begin{pmatrix} ra_{1,1} & \ldots & ra_{m,1} \\ \vdots & & \\ ra_{1,n} & \ldots & ra_{m,n} \end{pmatrix} + \begin{pmatrix} sb_{1,1} & \ldots & sb_{m,1} \\ \vdots & & \\ sb_{1,n} & \ldots & sb_{m,n} \end{pmatrix} \\
&= rA^{\text{trans}} + sB^{\text{trans}}
\end{aligned}$$

Two.III.3.31 **(a)** These reductions give different bases.

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \xrightarrow{2\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) An easy example is this.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

This is a less simplistic example.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 4 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

(c) Assume that A and B are matrices with equal row spaces. Construct a matrix C with the rows of A above the rows of B, and another matrix D with the rows of B above the rows of A.

$$C = \begin{pmatrix} A \\ B \end{pmatrix} \qquad D = \begin{pmatrix} B \\ A \end{pmatrix}$$

Observe that C and D are row-equivalent (via a sequence of row-swaps) and so Gauss-Jordan reduce to the same reduced echelon form matrix.

Because the row spaces are equal, the rows of B are linear combinations of the rows of A so Gauss-Jordan reduction on C simply turns the rows of B to zero rows and thus the nonzero rows of C are just the nonzero rows obtained by Gauss-Jordan reducing A. The same can be said for the matrix D — Gauss-Jordan reduction on D gives the same non-zero rows as are produced by reduction on B alone. Therefore, A yields the same nonzero rows as C, which yields the same nonzero rows as D, which yields the same nonzero rows as B.

Two.III.3.32 It cannot be bigger.

Two.III.3.33 The number of rows in a maximal linearly independent set cannot exceed the number of rows. A better bound (the bound that is, in general, the best possible) is the minimum of m and n, because the row rank equals the column rank.

Two.III.3.34 Because the rows of a matrix A are turned into the columns of A^{trans} the dimension of the row space of A equals the dimension of the column space of A^{trans}. But the dimension of the row space of A is the rank of A and the dimension of the column space of A^{trans} is the rank of A^{trans}. Thus the two ranks are equal.

Two.III.3.35 False. The first is a set of columns while the second is a set of rows.

This example, however,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad A^{\text{trans}} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

indicates that as soon as we have a formal meaning for "the same", we can apply it here:

$$\text{Columnspace}(A) = [\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}\}]$$

while

$$\text{Rowspace}(A^{\text{trans}}) = [\{\begin{pmatrix} 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 6 \end{pmatrix}\}]$$

are "the same" as each other.

Two.III.3.36 No. Here, Gauss' method does not change the column space.

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Two.III.3.37 A linear system

$$c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{d}$$

has a solution if and only if $\vec{d}$ is in the span of the set $\{\vec{a}_1, \ldots, \vec{a}_n\}$. That's true if and only if the column rank of the augmented matrix equals the column rank of the matrix of coefficients. Since rank equals the column rank, the system has a solution if and only if the rank of its augmented matrix equals the rank of its matrix of coefficients.

Two.III.3.38 **(a)** Row rank equals column rank so each is at most the minimum of the number of rows and columns. Hence both can be full only if the number of rows equals the number of columns. (Of course, the converse does not hold: a square matrix need not have full row rank or full column rank.)

(b) If A has full row rank then, no matter what the right-hand side, Gauss' method on the augmented matrix ends with a leading one in each row and none of those leading ones in the furthest right column (the "augmenting" column). Back substitution then gives a solution.

On the other hand, if the linear system lacks a solution for some right-hand side it can only be because Gauss' method leaves some row so that it is all zeroes to the left of the "augmenting" bar and has a nonzero entry on the right. Thus, if A does not have a solution for some right-hand sides, then A does not have full row rank because some of its rows have been eliminated.

(c) The matrix A has full column rank if and only if its columns form a linearly independent set. That's equivalent to the existence of only the trivial linear relationship among the columns, so the only solution of the system is where each variable is 0.

(d) The matrix A has full column rank if and only if the set of its columns is linearly independent, and so forms a basis for its span. That's equivalent to the existence of a unique linear representation of all vectors in that span. That proves it, since any linear representation of a vector in the span is a solution of the linear system.

Two.III.3.39 Instead of the row spaces being the same, the row space of B would be a subspace (possibly equal to) the row space of A.

Two.III.3.40 Clearly $\text{rank}(A) = \text{rank}(-A)$ as Gauss' method allows us to multiply all rows of a matrix by -1. In the same way, when $k \neq 0$ we have $\text{rank}(A) = \text{rank}(kA)$.

Addition is more interesting. The rank of a sum can be smaller than the rank of the summands.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The rank of a sum can be bigger than the rank of the summands.

$$\begin{pmatrix}1&2\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

But there is an upper bound (other than the size of the matrices). In general, $\text{rank}(A+B)\leq\text{rank}(A)+\text{rank}(B)$.

To prove this, note that Gaussian elimination can be performed on $A+B$ in either of two ways: we can first add A to B and then apply the appropriate sequence of reduction steps

$$(A+B)\xrightarrow{\text{step}_1}\cdots\xrightarrow{\text{step}_k}\text{echelon form}$$

or we can get the same results by performing step_1 through step_k separately on A and B, and then adding. The largest rank that we can end with in the second case is clearly the sum of the ranks. (The matrices above give examples of both possibilities, $\text{rank}(A+B)<\text{rank}(A)+\text{rank}(B)$ and $\text{rank}(A+B)=\text{rank}(A)+\text{rank}(B)$, happening.)

Subsection Two.III.4: Combining Subspaces

Two.III.4.20 With each of these we can apply Lemma 4.15.

(a) Yes. The plane is the sum of this W_1 and W_2 because for any scalars a and b

$$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a-b\\0\end{pmatrix}+\begin{pmatrix}b\\b\end{pmatrix}$$

shows that the general vector is a sum of vectors from the two parts. And, these two subspaces are (different) lines through the origin, and so have a trivial intersection.

(b) Yes. To see that any vector in the plane is a combination of vectors from these parts, consider this relationship.

$$\begin{pmatrix}a\\b\end{pmatrix}=c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\1.1\end{pmatrix}$$

We could now simply note that the set

$$\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1.1\end{pmatrix}\}$$

is a basis for the space (because it is clearly linearly independent, and has size two in $\mathbb{R}^2$), and thus ther is one and only one solution to the above equation, implying that all decompositions are unique. Alternatively, we can solve

$$\begin{array}{rl}c_1+c_2=a\\c_1+1.1c_2=b\end{array}\xrightarrow{-\rho_1+\rho_2}\begin{array}{rl}c_1+c_2=&a\\0.1c_2=&-a+b\end{array}$$

to get that $c_2=10(-a+b)$ and $c_1=11a-10b$, and so we have

$$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}11a-10b\\11a-10b\end{pmatrix}+\begin{pmatrix}-10a+10b\\1.1\cdot(-10a+10b)\end{pmatrix}$$

as required. As with the prior answer, each of the two subspaces is a line through the origin, and their intersection is trivial.

(c) Yes. Each vector in the plane is a sum in this way

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$$

and the intersection of the two subspaces is trivial.

(d) No. The intersection is not trivial.

(e) No. These are not subspaces.

Two.III.4.21 With each of these we can use Lemma 4.15.

(a) Any vector in $\mathbb{R}^3$ can be decomposed as this sum.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\0\end{pmatrix}+\begin{pmatrix}0\\0\\z\end{pmatrix}$$

And, the intersection of the xy-plane and the z-axis is the trivial subspace.