Chapter One: Linear Systems

Subsection One.I.1: Gauss' Method

One.I.1.16 Gauss' method can be performed in different ways, so these simply exhibit one possible way to get the answer.

(a) Gauss' method

$$\xrightarrow{-(1/2)\rho_1+\rho_2}\quad \begin{array}{rcrcr} 2x & + & 3y & = & 13 \\ & - & (5/2)y & = & -15/2 \end{array}$$

gives that the solution is $y = 3$ and $x = 2$.

(b) Gauss' method here

$$\xrightarrow[\rho_1+\rho_3]{-3\rho_1+\rho_2}\quad \begin{array}{rcrcr} x & & -z & = & 0 \\ & y & +3z & = & 1 \\ & y & & = & 4 \end{array} \quad \xrightarrow{-\rho_2+\rho_3}\quad \begin{array}{rcrcr} x & & -z & = & 0 \\ & y & +3z & = & 1 \\ & & -3z & = & 3 \end{array}$$

gives $x = -1$, $y = 4$, and $z = -1$.

One.I.1.17 **(a)** Gaussian reduction

$$\xrightarrow{-(1/2)\rho_1+\rho_2}\quad \begin{array}{rcrcr} 2x & + & 2y & = & 5 \\ & & -5y & = & -5/2 \end{array}$$

shows that $y = 1/2$ and $x = 2$ is the unique solution.

(b) Gauss' method

$$\xrightarrow{\rho_1+\rho_2}\quad \begin{array}{rcrcr} -x & + & y & = & 1 \\ & & 2y & = & 3 \end{array}$$

gives $y = 3/2$ and $x = 1/2$ as the only solution.

(c) Row reduction

$$\xrightarrow{-\rho_1+\rho_2}\quad \begin{array}{rcrcrcr} x & - & 3y & + & z & = & 1 \\ & & 4y & + & z & = & 13 \end{array}$$

shows, because the variable z is not a leading variable in any row, that there are many solutions.

(d) Row reduction

$$\xrightarrow{-3\rho_1+\rho_2}\quad \begin{array}{rcrcr} -x & - & y & = & 1 \\ & & 0 & = & -1 \end{array}$$

shows that there is no solution.

(e) Gauss' method

$$\xrightarrow{\rho_1\leftrightarrow\rho_4}\quad \begin{array}{rcrcrcr} x & + & y & - & z & = & 10 \\ 2x & - & 2y & + & z & = & 0 \\ x & & & + & z & = & 5 \\ & & 4y & + & z & = & 20 \end{array} \quad \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\quad \begin{array}{rcrcrcr} x & + & y & - & z & = & 10 \\ & & -4y & + & 3z & = & -20 \\ & & -y & + & 2z & = & -5 \\ & & 4y & + & z & = & 20 \end{array} \quad \xrightarrow[\rho_2+\rho_4]{-(1/4)\rho_2+\rho_3}\quad \begin{array}{rcrcrcr} x & + & y & - & z & = & 10 \\ & & -4y & + & 3z & = & -20 \\ & & & & (5/4)z & = & 0 \\ & & & & 4z & = & 0 \end{array}$$

gives the unique solution $(x, y, z) = (5, 5, 0)$.

(f) Here Gauss' method gives

$$\xrightarrow[-2\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3}\quad \begin{array}{rcrcrcr} 2x & + & z & + & w & = & 5 \\ y & & & - & w & = & -1 \\ & - & (5/2)z & - & (5/2)w & = & -15/2 \\ y & & & - & w & = & -1 \end{array} \quad \xrightarrow{-\rho_2+\rho_4}\quad \begin{array}{rcrcrcr} 2x & + & z & + & w & = & 5 \\ y & & & - & w & = & -1 \\ & - & (5/2)z & - & (5/2)w & = & -15/2 \\ & & & & 0 & = & 0 \end{array}$$

which shows that there are many solutions.

One.I.1.18 **(a)** From $x = 1 - 3y$ we get that $2(1 - 3y) + y = -3$, giving $y = 1$.

(b) From $x = 1 - 3y$ we get that $2(1 - 3y) + 2y = 0$, leading to the conclusion that $y = 1/2$.

Users of this method must check any potential solutions by substituting back into all the equations.

One.I.1.19 Do the reduction

$$\xrightarrow{-3\rho_1+\rho_2}\quad\begin{array}{rl} x-y= & 1 \\ 0= & -3+k \end{array}$$

to conclude this system has no solutions if $k\neq 3$ and if $k=3$ then it has infinitely many solutions. It never has a unique solution.

One.I.1.20 Let $x=\sin\alpha$, $y=\cos\beta$, and $z=\tan\gamma$:

$$\begin{array}{rl} 2x-y+3z= & 3 \\ 4x+2y-2z= & 10 \\ 6x-3y+z= & 9 \end{array}\quad\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\quad\begin{array}{rl} 2x-y+3z= & 3 \\ 4y-8z= & 4 \\ -8z= & 0 \end{array}$$

gives $z=0$, $y=1$, and $x=2$. Note that no α satisfies that requirement.

One.I.1.21 **(a)** Gauss' method

$$\xrightarrow[\substack{-\rho_1+\rho_3\\ -2\rho_1+\rho_4}]{-3\rho_1+\rho_2}\quad\begin{array}{rl} x-3y= & b_1 \\ 10y= & -3b_1+b_2 \\ 10y= & -b_1+b_3 \\ 10y= & -2b_1+b_4 \end{array}\quad\xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3}\quad\begin{array}{rl} x-3y= & b_1 \\ 10y= & -3b_1+b_2 \\ 0= & 2b_1-b_2+b_3 \\ 0= & b_1-b_2+b_4 \end{array}$$

shows that this system is consistent if and only if both $b_3=-2b_1+b_2$ and $b_4=-b_1+b_2$.

(b) Reduction

$$\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\quad\begin{array}{rl} x_1+2x_2+3x_3= & b_1 \\ x_2-3x_3= & -2b_1+b_2 \\ -2x_2+5x_3= & -b_1+b_3 \end{array}\quad\xrightarrow{2\rho_2+\rho_3}\quad\begin{array}{rl} x_1+2x_2+3x_3= & b_1 \\ x_2-3x_3= & -2b_1+b_2 \\ -x_3= & -5b_1++2b_2+b_3 \end{array}$$

shows that each of b_1, b_2, and b_3 can be any real number — this system always has a unique solution.

One.I.1.22 This system with more unknowns than equations

$$\begin{array}{rl} x+y+z= & 0 \\ x+y+z= & 1 \end{array}$$

has no solution.

One.I.1.23 Yes. For example, the fact that the same reaction can be performed in two different flasks shows that twice any solution is another, different, solution (if a physical reaction occurs then there must be at least one nonzero solution).

One.I.1.24 Because $f(1)=2$, $f(-1)=6$, and $f(2)=3$ we get a linear system.

$$\begin{array}{rl} 1a+1b+c= & 2 \\ 1a-1b+c= & 6 \\ 4a+2b+c= & 3 \end{array}$$

Gauss' method

$$\xrightarrow[-4\rho_1+\rho_2]{-\rho_1+\rho_2}\quad\begin{array}{rl} a+b+c= & 2 \\ -2b= & 4 \\ -2b-3c= & -5 \end{array}\quad\xrightarrow{-\rho_2+\rho_3}\quad\begin{array}{rl} a+b+c= & 2 \\ -2b= & 4 \\ -3c= & -9 \end{array}$$

shows that the solution is $f(x)=1x^2-2x+3$.

One.I.1.25 **(a)** Yes, by inspection the given equation results from $-\rho_1+\rho_2$.

(b) No. The given equation is satisfied by the pair $(1,1)$. However, that pair does not satisfy the first equation in the system.

(c) Yes. To see if the given row is $c_1\rho_1+c_2\rho_2$, solve the system of equations relating the coefficients of x, y, z, and the constants:

$$\begin{array}{rl} 2c_1+6c_2= & 6 \\ c_1-3c_2= & -9 \\ -c_1+c_2= & 5 \\ 4c_1+5c_2= & -2 \end{array}$$

and get $c_1=-3$ and $c_2=2$, so the given row is $-3\rho_1+2\rho_2$.

One.I.1.26 If $a\neq 0$ then the solution set of the first equation is $\{(x,y)\mid x=(c-by)/a\}$. Taking $y=0$ gives the solution $(c/a,0)$, and since the second equation is supposed to have the same solution set, substituting into it gives that $a(c/a)+d\cdot 0=e$, so $c=e$. Then taking $y=1$ in $x=(c-by)/a$ gives that $a((c-b)/a)+d\cdot 1=e$, which gives that $b=d$. Hence they are the same equation.

When $a=0$ the equations can be different and still have the same solution set: e.g., $0x+3y=6$ and $0x+6y=12$.

One.I.1.27 We take three cases: that $a \neq 0$, that $a = 0$ and $c \neq 0$, and that both $a = 0$ and $c = 0$.

For the first, we assume that $a \neq 0$. Then the reduction

$$\xrightarrow{-(c/a)\rho_1+\rho_2} \begin{array}{rcl} ax + & by = & j \\ & (-\frac{cb}{a} + d)y = & -\frac{cj}{a} + k \end{array}$$

shows that this system has a unique solution if and only if $-(cb/a) + d \neq 0$; remember that $a \neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a)+d = (ad-bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. Thus, in this first case, there is a unique solution if and only if $ad - bc \neq 0$.

In the second case, if $a = 0$ but $c \neq 0$, then we swap

$$\begin{array}{rl} cx + dy &= k \\ by &= j \end{array}$$

to conclude that the system has a unique solution if and only if $b \neq 0$ (we use the case assumption that $c \neq 0$ to get a unique x in back substitution). But — where $a = 0$ and $c \neq 0$ — the condition "$b \neq 0$" is equivalent to the condition "$ad - bc \neq 0$". That finishes the second case.

Finally, for the third case, if both a and c are 0 then the system

$$\begin{array}{rl} 0x + by &= j \\ 0x + dy &= k \end{array}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x, y) will do), but it never has a unique solution. Note that $a = 0$ and $c = 0$ gives that $ad - bc = 0$.

One.I.1.28 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.

Thus the lines can share one point (giving a unique solution), share no points (giving no solutions), or share at least two points (which makes them the same line).

One.I.1.29 For the reduction operation of multiplying ρ_i by a nonzero real number k, we have that $(s_1, \ldots, s_n)$ satisfies this system

$$\begin{array}{rl} a_{1,1}x_1 + \ a_{1,2}x_2 + \cdots + \ a_{1,n}x_n = & d_1 \\ & \vdots \\ ka_{i,1}x_1 + ka_{i,2}x_2 + \cdots + ka_{i,n}x_n = & kd_i \\ & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = & d_m \end{array}$$

if and only if

$$\begin{array}{rl} a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n = & d_1 \\ & \vdots \\ \text{and } ka_{i,1}s_1 + ka_{i,2}s_2 + \cdots + ka_{i,n}s_n = & kd_i \\ & \vdots \\ \text{and } a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = & d_m \end{array}$$

by the definition of 'satisfies'. But, because $k \neq 0$, that's true if and only if

$$\begin{array}{rl} a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n = & d_1 \\ & \vdots \\ \text{and } a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = & d_i \\ & \vdots \\ \text{and } a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = & d_m \end{array}$$

(this is straightforward cancelling on both sides of the i-th equation), which says that $(s_1, \ldots, s_n)$

solves

$$\begin{aligned} a_{1,1}x_1 + \ a_{1,2}x_2 + \cdots + \ a_{1,n}x_n &= \ d_1 \\ &\vdots \\ a_{i,1}x_1 + \ a_{i,2}x_2 + \cdots + \ a_{i,n}x_n &= \ d_i \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= d_m \end{aligned}$$

as required.

For the pivot operation $k\rho_i + \rho_j$, we have that $(s_1, \ldots, s_n)$ satisfies

$$\begin{aligned} a_{1,1}x_1 + \cdots + \qquad a_{1,n}x_n &= \qquad d_1 \\ &\vdots \\ a_{i,1}x_1 + \cdots + \qquad a_{i,n}x_n &= \qquad d_i \\ &\vdots \\ (ka_{i,1} + a_{j,1})x_1 + \cdots + (ka_{i,n} + a_{j,n})x_n &= kd_i + d_j \\ &\vdots \\ a_{m,1}x_1 + \cdots + \qquad a_{m,n}x_n &= d_m \end{aligned}$$

if and only if

$$\begin{aligned} a_{1,1}s_1 + \cdots + a_{1,n}s_n &= d_1 \\ &\vdots \\ \text{and } a_{i,1}s_1 + \cdots + a_{i,n}s_n &= d_i \\ &\vdots \\ \text{and } (ka_{i,1} + a_{j,1})s_1 + \cdots + (ka_{i,n} + a_{j,n})s_n &= kd_i + d_j \\ &\vdots \\ \text{and } a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n &= d_m \end{aligned}$$

again by the definition of 'satisfies'. Subtract k times the i-th equation from the j-th equation (remark: here is where $i \neq j$ is needed; if $i = j$ then the two d_i's above are not equal) to get that the previous compound statement holds if and only if

$$\begin{aligned} a_{1,1}s_1 + \cdots + a_{1,n}s_n &= d_1 \\ &\vdots \\ \text{and } a_{i,1}s_1 + \cdots + a_{i,n}s_n &= d_i \\ &\vdots \\ \text{and } (ka_{i,1} + a_{j,1})s_1 + \cdots + (ka_{i,n} + a_{j,n})s_n & \\ - (ka_{i,1}s_1 + \cdots + ka_{i,n}s_n) &= kd_i + d_j - kd_i \\ &\vdots \\ \text{and } a_{m,1}s_1 + \cdots + a_{m,n}s_n &= d_m \end{aligned}$$

which, after cancellation, says that $(s_1, \ldots, s_n)$ solves

$$\begin{aligned} a_{1,1}x_1 + \cdots + \ a_{1,n}x_n &= \ d_1 \\ &\vdots \\ a_{i,1}x_1 + \cdots + \ a_{i,n}x_n &= \ d_i \\ &\vdots \\ a_{j,1}x_1 + \cdots + \ a_{j,n}x_n &= \ d_j \\ &\vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n &= d_m \end{aligned}$$

as required.

One.I.1.30 Yes, this one-equation system:

$$0x + 0y = 0$$

is satisfied by every $(x, y) \in \mathbb{R}^2$.

One.I.1.31 Yes. This sequence of operations swaps rows i and j

$$\xrightarrow{\rho_i+\rho_j}\quad\xrightarrow{-\rho_j+\rho_i}\quad\xrightarrow{\rho_i+\rho_j}\quad\xrightarrow{-1\rho_i}$$

so the row-swap operation is redundant in the presence of the other two.

One.I.1.32 Swapping rows is reversed by swapping back.

$$\begin{array}{rcl} a_{1,1}x_1+\cdots+\ a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{m,1}x_1+\cdots+a_{m,n}x_n &=& d_m \end{array} \xrightarrow{\rho_i\leftrightarrow\rho_j}\ \xrightarrow{\rho_j\leftrightarrow\rho_i} \begin{array}{rcl} a_{1,1}x_1+\cdots+\ a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{m,1}x_1+\cdots+a_{m,n}x_n &=& d_m \end{array}$$

Multiplying both sides of a row by $k\neq 0$ is reversed by dividing by k.

$$\begin{array}{rcl} a_{1,1}x_1+\cdots+\ a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{m,1}x_1+\cdots+a_{m,n}x_n &=& d_m \end{array} \xrightarrow{k\rho_i}\ \xrightarrow{(1/k)\rho_i} \begin{array}{rcl} a_{1,1}x_1+\cdots+\ a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{m,1}x_1+\cdots+a_{m,n}x_n &=& d_m \end{array}$$

Adding k times a row to another is reversed by adding $-k$ times that row.

$$\begin{array}{rcl} a_{1,1}x_1+\cdots+\ a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{m,1}x_1+\cdots+a_{m,n}x_n &=& d_m \end{array} \xrightarrow{k\rho_i+\rho_j}\ \xrightarrow{-k\rho_i+\rho_j} \begin{array}{rcl} a_{1,1}x_1+\cdots+\ a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{m,1}x_1+\cdots+a_{m,n}x_n &=& d_m \end{array}$$

Remark: observe for the third case that if we were to allow $i=j$ then the result wouldn't hold.

$$3x+2y=7 \xrightarrow{2\rho_1+\rho_1} 9x+6y=21 \xrightarrow{-2\rho_1+\rho_1} -9x-6y=-21$$

One.I.1.33 Let p, n, and d be the number of pennies, nickels, and dimes. For variables that are real numbers, this system

$$\begin{array}{rcl} p+\ \ n+\ \ \ d &=& 13 \\ p+5n+10d &=& 83 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} p+\ \ n+\ \ d &=& 13 \\ 4n+9d &=& 70 \end{array}$$

has infinitely many solutions. However, it has a limited number of solutions in which p, n, and d are non-negative integers. Running through $d=0$, …, $d=8$ shows that $(p,n,d)=(3,4,6)$ is the only sensible solution.

One.I.1.34 Solving the system

$$\begin{array}{rcl} (1/3)(a+b+c)+d &=& 29 \\ (1/3)(b+c+d)+a &=& 23 \\ (1/3)(c+d+a)+b &=& 21 \\ (1/3)(d+a+b)+c &=& 17 \end{array}$$

we obtain $a=12$, $b=9$, $c=3$, $d=21$. Thus the second item, 21, is the correct answer.

One.I.1.35 *This is how the answer was given in the cited source.* A comparison of the units and hundreds columns of this addition shows that there must be a carry from the tens column. The tens column then tells us that $A<H$, so there can be no carry from the units or hundreds columns. The five columns then give the following five equations.

$$\begin{aligned} A+E &= W \\ 2H &= A+10 \\ H &= W+1 \\ H+T &= E+10 \\ A+1 &= T \end{aligned}$$

The five linear equations in five unknowns, if solved simultaneously, produce the unique solution: $A=4$, $T=5$, $H=7$, $W=6$ and $E=2$, so that the original example in addition was $47474+5272=52746$.

One.I.1.36 *This is how the answer was given in the cited source.* Eight commissioners voted for B. To see this, we will use the given information to study how many voters chose each order of A, B, C.

The six orders of preference are ABC, ACB, BAC, BCA, CAB, CBA; assume they receive a, b, c, d, e, f votes respectively. We know that

$$\begin{aligned} a+b+e &= 11 \\ d+e+f &= 12 \\ a+c+d &= 14 \end{aligned}$$

from the number preferring A over B, the number preferring C over A, and the number preferring B over C. Because 20 votes were cast, we also know that

$$\begin{aligned} c+d+f&=9\\ a+b+c&=8\\ b+e+f&=6 \end{aligned}$$

from the preferences for B over A, for A over C, and for C over B.

The solution is $a=6$, $b=1$, $c=1$, $d=7$, $e=4$, and $f=1$. The number of commissioners voting for B as their first choice is therefore $c+d=1+7=8$.

Comments. The answer to this question would have been the same had we known only that *at least* 14 commissioners preferred B over C.

The seemingly paradoxical nature of the commissioners's preferences (A is preferred to B, and B is preferred to C, and C is preferred to A), an example of "non-transitive dominance", is not uncommon when individual choices are pooled.

One.I.1.37 *This is how the answer was given in the cited source. We have not used "dependent" yet; it means here that Gauss' method shows that there is not a unique solution.* If $n\geq 3$ the system is dependent and the solution is not unique. Hence $n<3$. But the term "system" implies $n>1$. Hence $n=2$. If the equations are

$$\begin{aligned} ax+\ (a+d)y&=a+2d\\ (a+3d)x+(a+4d)y&=a+5d \end{aligned}$$

then $x=-1$, $y=2$.

Subsection One.I.2: Describing the Solution Set

One.I.2.15 **(a)** 2 **(b)** 3 **(c)** -1 **(d)** Not defined.

One.I.2.16 **(a)** 2×3 **(b)** 3×2 **(c)** 2×2

One.I.2.17 **(a)** $\begin{pmatrix}5\\1\\5\end{pmatrix}$ **(b)** $\begin{pmatrix}20\\-5\end{pmatrix}$ **(c)** $\begin{pmatrix}-2\\4\\0\end{pmatrix}$ **(d)** $\begin{pmatrix}41\\52\end{pmatrix}$ **(e)** Not defined. **(f)** $\begin{pmatrix}12\\8\\4\end{pmatrix}$

One.I.2.18 **(a)** This reduction

$$\left(\begin{array}{cc|c}3&6&18\\1&2&6\end{array}\right)\xrightarrow{(-1/3)\rho_1+\rho_2}\left(\begin{array}{cc|c}3&6&18\\0&0&0\end{array}\right)$$

leaves x leading and y free. Making y the parameter, we have $x=6-2y$ so the solution set is

$$\{\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}y \mid y\in\mathbb{R}\}.$$

(b) This reduction

$$\left(\begin{array}{cc|c}1&1&1\\1&-1&-1\end{array}\right)\xrightarrow{-\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&1\\0&-2&-2\end{array}\right)$$

gives the unique solution $y=1$, $x=0$. The solution set is

$$\{\begin{pmatrix}0\\1\end{pmatrix}\}.$$

(c) This use of Gauss' method

$$\left(\begin{array}{ccc|c}1&0&1&4\\1&-1&2&5\\4&-1&5&17\end{array}\right)\xrightarrow[-4\rho_1+\rho_3]{-\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&0&1&4\\0&-1&1&1\\0&-1&1&1\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&0&1&4\\0&-1&1&1\\0&0&0&0\end{array}\right)$$

leaves x_1 and x_2 leading with x_3 free. The solution set is

$$\{\begin{pmatrix}4\\-1\\0\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3 \mid x_3\in\mathbb{R}\}.$$

(d) This reduction

$$\left(\begin{array}{ccc|c}2&1&-1&2\\2&0&1&3\\1&-1&0&0\end{array}\right)\xrightarrow[-(1/2)\rho_1+\rho_3]{-\rho_1+\rho_2}\left(\begin{array}{ccc|c}2&1&-1&2\\0&-1&2&1\\0&-3/2&1/2&-1\end{array}\right)\xrightarrow{(-3/2)\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&1&-1&2\\0&-1&2&1\\0&0&-5/2&-5/2\end{array}\right)$$

shows that the solution set is a singleton set.

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}$$

(e) This reduction is easy

$$\left(\begin{array}{cccc|c}1&2&-1&0&3\\2&1&0&1&4\\1&-1&1&1&1\end{array}\right)\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&2&-1&0&3\\0&-3&2&1&-2\\0&-3&2&1&-2\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&2&-1&0&3\\0&-3&2&1&-2\\0&0&0&0&0\end{array}\right)$$

and ends with x and y leading, while z and w are free. Solving for y gives $y=(2+2z+w)/3$ and substitution shows that $x+2(2+2z+w)/3-z=3$ so $x=(5/3)-(1/3)z-(2/3)w$, making the solution set

$$\{\begin{pmatrix}5/3\\2/3\\0\\0\end{pmatrix}+\begin{pmatrix}-1/3\\2/3\\1\\0\end{pmatrix}z+\begin{pmatrix}-2/3\\1/3\\0\\1\end{pmatrix}w\bigm| z,w\in\mathbb{R}\}.$$

(f) The reduction

$$\left(\begin{array}{cccc|c}1&0&1&1&4\\2&1&0&-1&2\\3&1&1&0&7\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&0&1&1&4\\0&1&-2&-3&-6\\0&1&-2&-3&-5\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&0&1&1&4\\0&1&-2&-3&-6\\0&0&0&0&1\end{array}\right)$$

shows that there is no solution — the solution set is empty.

One.I.2.19 **(a)** This reduction

$$\left(\begin{array}{ccc|c}2&1&-1&1\\4&-1&0&3\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{ccc|c}2&1&-1&1\\0&-3&2&1\end{array}\right)$$

ends with x and y leading while z is free. Solving for y gives $y=(1-2z)/(-3)$, and then substitution $2x+(1-2z)/(-3)-z=1$ shows that $x=((4/3)+(1/3)z)/2$. Hence the solution set is

$$\{\begin{pmatrix}2/3\\-1/3\\0\end{pmatrix}+\begin{pmatrix}1/6\\2/3\\1\end{pmatrix}z\bigm| z\in\mathbb{R}\}.$$

(b) This application of Gauss' method

$$\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&-1&3\\1&2&3&-1&7\end{array}\right)\xrightarrow{-\rho_1+\rho_3}\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&-1&3\\0&2&4&-1&6\end{array}\right)\xrightarrow{-2\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&-1&3\\0&0&0&1&0\end{array}\right)$$

leaves x, y, and w leading. The solution set is

$$\{\begin{pmatrix}1\\3\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}z\bigm| z\in\mathbb{R}\}.$$

(c) This row reduction

$$\left(\begin{array}{cccc|c}1&-1&1&0&0\\0&1&0&1&0\\3&-2&3&1&0\\0&-1&0&-1&0\end{array}\right)\xrightarrow{-3\rho_1+\rho_3}\left(\begin{array}{cccc|c}1&-1&1&0&0\\0&1&0&1&0\\0&1&0&1&0\\0&-1&0&-1&0\end{array}\right)\xrightarrow[\rho_2+\rho_4]{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&-1&1&0&0\\0&1&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)$$

ends with z and w free. The solution set is

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w\bigm| z,w\in\mathbb{R}\}.$$

(d) Gauss' method done in this way

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -1 & 1 \\ 3 & -1 & 1 & 1 & 1 & 3 \end{array}\right) \xrightarrow{-3\rho_1+\rho_2} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & -1 & 1 \\ 0 & -7 & -8 & -2 & 4 & 0 \end{array}\right)$$

ends with c, d, and e free. Solving for b shows that $b = (8c + 2d - 4e)/(-7)$ and then substitution $a + 2(8c + 2d - 4e)/(-7) + 3c + 1d - 1e = 1$ shows that $a = 1 - (5/7)c - (3/7)d - (1/7)e$ and so the solution set is

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -5/7 \\ -8/7 \\ 1 \\ 0 \\ 0 \end{pmatrix} c + \begin{pmatrix} -3/7 \\ -2/7 \\ 0 \\ 1 \\ 0 \end{pmatrix} d + \begin{pmatrix} -1/7 \\ 4/7 \\ 0 \\ 0 \\ 1 \end{pmatrix} e \bigm| c, d, e \in \mathbb{R}\}.$$

One.I.2.20 For each problem we get a system of linear equations by looking at the equations of components.

(a) $k = 5$

(b) The second components show that $i = 2$, the third components show that $j = 1$.

(c) $m = -4$, $n = 2$

One.I.2.21 For each problem we get a system of linear equations by looking at the equations of components.

(a) Yes; take $k = -1/2$.

(b) No; the system with equations $5 = 5 \cdot j$ and $4 = -4 \cdot j$ has no solution.

(c) Yes; take $r = 2$.

(d) No. The second components give $k = 0$. Then the third components give $j = 1$. But the first components don't check.

One.I.2.22 This system has 1 equation. The leading variable is x_1, the other variables are free.

$$\{\begin{pmatrix} -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} x_2 + \cdots + \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 1 \end{pmatrix} x_n \bigm| x_2, \ldots, x_n \in \mathbb{R}\}$$

One.I.2.23 **(a)** Gauss' method here gives

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & a \\ 2 & 0 & 1 & 0 & b \\ 1 & 1 & 0 & 2 & c \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & a \\ 0 & -4 & 1 & 2 & -2a+b \\ 0 & -1 & 0 & 3 & -a+c \end{array}\right)$$

$$\xrightarrow{-(1/4)\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & a \\ 0 & -4 & 1 & 2 & -2a+b \\ 0 & 0 & -1/4 & 5/2 & -(1/2)a-(1/4)b+c \end{array}\right),$$

leaving w free. Solve: $z = 2a + b - 4c + 10w$, and $-4y = -2a + b - (2a + b - 4c + 10w) - 2w$ so $y = a - c + 3w$, and $x = a - 2(a - c + 3w) + w = -a + 2c - 5w$. Therefore the solution set is this.

$$\{\begin{pmatrix} -a+2c \\ a-c \\ 2a+b-4c \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \\ 10 \\ 1 \end{pmatrix} w \bigm| w \in \mathbb{R}\}$$

(b) Plug in with $a = 3$, $b = 1$, and $c = -2$.

$$\{\begin{pmatrix} -7 \\ 5 \\ 15 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \\ 10 \\ 1 \end{pmatrix} w \bigm| w \in \mathbb{R}\}$$

One.I.2.24 Leaving the comma out, say by writing a_{123}, is ambiguous because it could mean $a_{1,23}$ or $a_{12,3}$.

One.I.2.25 **(a)** $\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$

One.I.2.26 **(a)** $\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$ **(b)** $\begin{pmatrix}2&1\\-3&1\end{pmatrix}$ **(c)** $\begin{pmatrix}5&10\\10&5\end{pmatrix}$ **(d)** $\begin{pmatrix}1&1&0\end{pmatrix}$

One.I.2.27 **(a)** Plugging in $x=1$ and $x=-1$ gives

$$\begin{array}{rl}a+b+c=2\\a-b+c=6\end{array}\xrightarrow{-\rho_1+\rho_2}\begin{array}{rl}a+b+c=2\\-2b=4\end{array}$$

so the set of functions is $\{f(x)=(4-c)x^2-2x+c \mid c\in\mathbb{R}\}$.

(b) Putting in $x=1$ gives

$$a+b+c=2$$

so the set of functions is $\{f(x)=(2-b-c)x^2+bx+c \mid b,c\in\mathbb{R}\}$.

One.I.2.28 On plugging in the five pairs (x,y) we get a system with the five equations and six unknowns a, ..., f. Because there are more unknowns than equations, if no inconsistency exists among the equations then there are infinitely many solutions (at least one variable will end up free).

But no inconsistency can exist because $a=0$, ..., $f=0$ is a solution (we are only using this zero solution to show that the system is consistent — the prior paragraph shows that there are nonzero solutions).

One.I.2.29 **(a)** Here is one — the fourth equation is redundant but still OK.

$$\begin{array}{rl}x+y-z+w&=0\\y-z&=0\\2z+2w&=0\\z+w&=0\end{array}$$

(b) Here is one.

$$\begin{array}{rl}x+y-z+w&=0\\w&=0\\w&=0\\w&=0\end{array}$$

(c) This is one.

$$\begin{array}{rl}x+y-z+w&=0\\x+y-z+w&=0\\x+y-z+w&=0\\x+y-z+w&=0\end{array}$$

One.I.2.30 *This is how the answer was given in the cited source.*

(a) Formal solution of the system yields

$$x=\frac{a^3-1}{a^2-1}\qquad y=\frac{-a^2+a}{a^2-1}.$$

If $a+1\neq 0$ and $a-1\neq 0$, then the system has the single solution

$$x=\frac{a^2+a+1}{a+1}\qquad y=\frac{-a}{a+1}.$$

If $a=-1$, or if $a=+1$, then the formulas are meaningless; in the first instance we arrive at the system

$$\begin{cases}-x+y=1\\ x-y=1\end{cases}$$

which is a contradictory system. In the second instance we have

$$\begin{cases}x+y=1\\x+y=1\end{cases}$$

which has an infinite number of solutions (for example, for x arbitrary, $y=1-x$).

(b) Solution of the system yields

$$x=\frac{a^4-1}{a^2-1}\qquad y=\frac{-a^3+a}{a^2-1}.$$

Here, is $a^2-1\neq 0$, the system has the single solution $x=a^2+1$, $y=-a$. For $a=-1$ and $a=1$, we obtain the systems

$$\begin{cases}-x+y=-1\\ x-y=1\end{cases}\qquad\begin{cases}x+y=1\\x+y=1\end{cases}$$

both of which have an infinite number of solutions.

One.I.2.31 *This is how the answer was given in the cited source.* Let u, v, x, y, z be the volumes in cm^3 of Al, Cu, Pb, Ag, and Au, respectively, contained in the sphere, which we assume to be not hollow. Since the loss of weight in water (specific gravity 1.00) is 1000 grams, the volume of the sphere is 1000 cm^3. Then the data, some of which is superfluous, though consistent, leads to only 2 independent equations, one relating volumes and the other, weights.

$$\begin{aligned} u+\ \ v+\ \ x+\ \ y+\ \ z&=1000\\ 2.7u+8.9v+11.3x+10.5y+19.3z&=7558 \end{aligned}$$

Clearly the sphere must contain some aluminum to bring its mean specific gravity below the specific gravities of all the other metals. There is no unique result to this part of the problem, for the amounts of three metals may be chosen arbitrarily, provided that the choices will not result in negative amounts of any metal.

If the ball contains only aluminum and gold, there are 294.5 cm^3 of gold and 705.5 cm^3 of aluminum. Another possibility is 124.7 cm^3 each of Cu, Au, Pb, and Ag and 501.2 cm^3 of Al.

Subsection One.I.3: General = Particular + Homogeneous

One.I.3.15 For the arithmetic to these, see the answers from the prior subsection.

(a) The solution set is

$$\{\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}y \mid y\in\mathbb{R}\}.$$

Here the particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}6\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-2\\1\end{pmatrix}y \mid y\in\mathbb{R}\}.$$

(b) The solution set is

$$\{\begin{pmatrix}0\\1\end{pmatrix}\}.$$

The particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}0\\1\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}0\\0\end{pmatrix}\}$$

(c) The solution set is

$$\{\begin{pmatrix}4\\-1\\0\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3 \mid x_3\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}4\\-1\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3 \mid x_3\in\mathbb{R}\}.$$

(d) The solution set is a singleton

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}0\\0\\0\end{pmatrix}t \mid t\in\mathbb{R}\}.$$

(e) The solution set is

$$\{\begin{pmatrix}5/3\\2/3\\0\\0\end{pmatrix}+\begin{pmatrix}-1/3\\2/3\\1\\0\end{pmatrix}z+\begin{pmatrix}-2/3\\1/3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}5/3\\2/3\\0\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-1/3\\2/3\\1\\0\end{pmatrix}z+\begin{pmatrix}-2/3\\1/3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

(f) This system's solution set is empty. Thus, there is no particular solution. The solution set of the associated homogeneous system is

$$\{\begin{pmatrix}-1\\2\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\3\\0\\1\end{pmatrix}w \bigm| z,w\in\mathbb{R}\}.$$

One.I.3.16 The answers from the prior subsection show the row operations.

(a) The solution set is

$$\{\begin{pmatrix}2/3\\-1/3\\0\end{pmatrix}+\begin{pmatrix}1/6\\2/3\\1\end{pmatrix}z \bigm| z\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}2/3\\-1/3\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}1/6\\2/3\\1\end{pmatrix}z \bigm| z\in\mathbb{R}\}.$$

(b) The solution set is

$$\{\begin{pmatrix}1\\3\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}z \bigm| z\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}1\\3\\0\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}z \bigm| z\in\mathbb{R}\}.$$

(c) The solution set is

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w \bigm| z,w\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w \bigm| z,w\in\mathbb{R}\}.$$

(d) The solution set is

$$\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-5/7\\-8/7\\1\\0\\0\end{pmatrix}c+\begin{pmatrix}-3/7\\-2/7\\0\\1\\0\end{pmatrix}d+\begin{pmatrix}-1/7\\4/7\\0\\0\\1\end{pmatrix}e \bigm| c,d,e\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-5/7\\-8/7\\1\\0\\0\end{pmatrix}c+\begin{pmatrix}-3/7\\-2/7\\0\\1\\0\end{pmatrix}d+\begin{pmatrix}-1/7\\4/7\\0\\0\\1\end{pmatrix}e \bigm| c,d,e\in\mathbb{R}\}.$$

One.I.3.17 Just plug them in and see if they satisfy all three equations.

(a) No.
(b) Yes.
(c) Yes.

One.I.3.18 Gauss' method on the associated homogeneous system gives

$$\left(\begin{array}{cccc|c}1&-1&0&1&0\\2&3&-1&0&0\\0&1&1&1&0\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&-1&0&1&0\\0&5&-1&-2&0\\0&1&1&1&0\end{array}\right)\xrightarrow{-(1/5)\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&-1&0&1&0\\0&5&-1&-2&0\\0&0&6/5&7/5&0\end{array}\right)$$

so this is the solution to the homogeneous problem:

$$\{\begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix}w \bigm| w\in\mathbb{R}\}.$$

(a) That vector is indeed a particular solution, so the required general solution is

$$\{\begin{pmatrix}0\\0\\0\\4\end{pmatrix}+\begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix}w \bigm| w\in\mathbb{R}\}.$$

(b) That vector is a particular solution so the required general solution is

$$\{\begin{pmatrix}-5\\1\\-7\\10\end{pmatrix}+\begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix}w \bigm| w\in\mathbb{R}\}.$$

(c) That vector is not a solution of the system since it does not satisfy the third equation. No such general solution exists.

One.I.3.19 The first is nonsingular while the second is singular. Just do Gauss' method and see if the echelon form result has non-0 numbers in each entry on the diagonal.

One.I.3.20 **(a)** Nonsingular:

$$\xrightarrow{-\rho_1+\rho_2}\begin{pmatrix}1&2\\0&1\end{pmatrix}$$

ends with each row containing a leading entry.

(b) Singular:

$$\xrightarrow{3\rho_1+\rho_2}\begin{pmatrix}1&2\\0&0\end{pmatrix}$$

ends with row 2 without a leading entry.

(c) Neither. A matrix must be square for either word to apply.

(d) Singular.

(e) Nonsingular.

One.I.3.21 In each case we must decide if the vector is a linear combination of the vectors in the set.

(a) Yes. Solve

$$c_1\begin{pmatrix}1\\4\end{pmatrix}+c_2\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$$

with

$$\left(\begin{array}{cc|c}1&1&2\\4&5&3\end{array}\right)\xrightarrow{-4\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&1&-5\end{array}\right)$$

to conclude that there are c_1 and c_2 giving the combination.

(b) No. The reduction

$$\left(\begin{array}{cc|c}2&1&-1\\1&0&0\\0&1&1\end{array}\right)\xrightarrow{-(1/2)\rho_1+\rho_2}\left(\begin{array}{cc|c}2&1&-1\\0&-1/2&1/2\\0&1&1\end{array}\right)\xrightarrow{2\rho_2+\rho_3}\left(\begin{array}{cc|c}2&1&-1\\0&-1/2&1/2\\0&0&2\end{array}\right)$$

shows that

$$c_1\begin{pmatrix}2\\1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

has no solution.

(c) Yes. The reduction

$$\left(\begin{array}{cccc|c}1&2&3&4&1\\0&1&3&2&3\\4&5&0&1&0\end{array}\right)\xrightarrow{-4\rho_1+\rho_3}\left(\begin{array}{cccc|c}1&2&3&4&1\\0&1&3&2&3\\0&-3&-12&-15&-4\end{array}\right)\xrightarrow{3\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&2&3&4&1\\0&1&3&2&3\\0&0&-3&-9&5\end{array}\right)$$

shows that there are infinitely many ways

$$\{\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}-10\\8\\-5/3\\0\end{pmatrix}+\begin{pmatrix}-9\\7\\-3\\1\end{pmatrix}c_4 \bigm| c_4\in\mathbb{R}\}$$

to write

$$\begin{pmatrix}1\\3\\0\end{pmatrix}=c_1\begin{pmatrix}1\\0\\4\end{pmatrix}+c_2\begin{pmatrix}2\\1\\5\end{pmatrix}+c_3\begin{pmatrix}3\\3\\0\end{pmatrix}+c_4\begin{pmatrix}4\\2\\1\end{pmatrix}.$$

(d) No. Look at the third components.

One.I.3.22 Because the matrix of coefficients is nonsingular, Gauss' method ends with an echelon form where each variable leads an equation. Back substitution gives a unique solution.

(Another way to see the solution is unique is to note that with a nonsingular matrix of coefficients the associated homogeneous system has a unique solution, by definition. Since the general solution is the sum of a particular solution with each homogeneous solution, the general solution has (at most) one element.)

One.I.3.23 In this case the solution set is all of $\mathbb{R}^n$, and can be expressed in the required form

$$\{c_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}+\cdots+c_n\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \bigm| c_1,\ldots,c_n\in\mathbb{R}\}.$$

One.I.3.24 Assume $\vec{s},\vec{t}\in\mathbb{R}^n$ and write

$$\vec{s}=\begin{pmatrix}s_1\\\vdots\\s_n\end{pmatrix}\quad\text{and}\quad\vec{t}=\begin{pmatrix}t_1\\\vdots\\t_n\end{pmatrix}.$$

Also let $a_{i,1}x_1+\cdots+a_{i,n}x_n=0$ be the i-th equation in the homogeneous system.

(a) The check is easy:

$$\begin{aligned}a_{i,1}(s_1+t_1)+\cdots+a_{i,n}(s_n+t_n) &= (a_{i,1}s_1+\cdots+a_{i,n}s_n)+(a_{i,1}t_1+\cdots+a_{i,n}t_n)\\ &= 0+0.\end{aligned}$$

(b) This one is similar:

$$a_{i,1}(3s_1)+\cdots+a_{i,n}(3s_n)=3(a_{i,1}s_1+\cdots+a_{i,n}s_n)=3\cdot 0=0.$$

(c) This one is not much harder:

$$\begin{aligned}a_{i,1}(ks_1+mt_1)+\cdots+a_{i,n}(ks_n+mt_n) &= k(a_{i,1}s_1+\cdots+a_{i,n}s_n)+m(a_{i,1}t_1+\cdots+a_{i,n}t_n)\\ &= k\cdot 0+m\cdot 0.\end{aligned}$$

What is wrong with that argument is that any linear combination of the zero vector yields the zero vector again.

One.I.3.25 First the proof.

Gauss' method will use only rationals (e.g., $-(m/n)\rho_i+\rho_j$). Thus the solution set can be expressed using only rational numbers as the components of each vector. Now the particular solution is all rational.

There are infinitely many (rational vector) solutions if and only if the associated homogeneous system has infinitely many (real vector) solutions. That's because setting any parameters to be rationals will produce an all-rational solution.

Subsection One.II.1: Vectors in Space

One.II.1.1 **(a)** $\begin{pmatrix}2\\1\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\2\end{pmatrix}$ **(c)** $\begin{pmatrix}4\\0\\-3\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$

One.II.1.2 **(a)** No, their canonical positions are different.

$$\begin{pmatrix}1\\-1\end{pmatrix} \qquad \begin{pmatrix}0\\3\end{pmatrix}$$

(b) Yes, their canonical positions are the same.

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

One.II.1.3 That line is this set.

$$\{\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}+\begin{pmatrix}7\\9\\-2\\4\end{pmatrix}t \bigm| t\in\mathbb{R}\}$$

Note that this system

$$\begin{aligned}-2+7t&=1\\ 1+9t&=0\\ 1-2t&=2\\ 0+4t&=1\end{aligned}$$

has no solution. Thus the given point is not in the line.

One.II.1.4 **(a)** Note that

$$\begin{pmatrix}2\\2\\2\\0\end{pmatrix}-\begin{pmatrix}1\\1\\5\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\-3\\1\end{pmatrix} \qquad \begin{pmatrix}3\\1\\0\\4\end{pmatrix}-\begin{pmatrix}1\\1\\5\\-1\end{pmatrix}=\begin{pmatrix}2\\0\\-5\\5\end{pmatrix}$$

and so the plane is this set.

$$\{\begin{pmatrix}1\\1\\5\\-1\end{pmatrix}+\begin{pmatrix}1\\1\\-3\\1\end{pmatrix}t+\begin{pmatrix}2\\0\\-5\\5\end{pmatrix}s \bigm| t,s\in\mathbb{R}\}$$

(b) No; this system

$$\begin{aligned}1+1t+2s&=0\\ 1+1t\phantom{{}+2s}&=0\\ 5-3t-5s&=0\\ -1+1t+5s&=0\end{aligned}$$

has no solution.

One.II.1.5 The vector

$$\begin{pmatrix}2\\0\\3\end{pmatrix}$$

is not in the line. Because

$$\begin{pmatrix}2\\0\\3\end{pmatrix}-\begin{pmatrix}-1\\0\\-4\end{pmatrix}=\begin{pmatrix}3\\0\\7\end{pmatrix}$$

that plane can be described in this way.

$$\{\begin{pmatrix}-1\\0\\-4\end{pmatrix}+m\begin{pmatrix}1\\1\\2\end{pmatrix}+n\begin{pmatrix}3\\0\\7\end{pmatrix} \bigm| m,n\in\mathbb{R}\}$$

One.II.1.6 The points of coincidence are solutions of this system.

$$\begin{aligned}t\phantom{{}+3s}&=1+2m\\ t+s&=1+3k\\ t+3s&=4m\end{aligned}$$

Gauss' method

$$\left(\begin{array}{cccc|c}1&0&0&-2&1\\1&1&-3&0&1\\1&3&0&-4&0\end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{cccc|c}1&0&0&-2&1\\0&1&-3&2&0\\0&3&0&-2&-1\end{array}\right) \xrightarrow{-3\rho_2+\rho_3} \left(\begin{array}{cccc|c}1&0&0&-2&1\\0&1&-3&2&0\\0&0&9&-8&-1\end{array}\right)$$

gives $k=-(1/9)+(8/9)m$, so $s=-(1/3)+(2/3)m$ and $t=1+2m$. The intersection is this.

$$\{\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\3\\0\end{pmatrix}(-\tfrac{1}{9}+\tfrac{8}{9}m)+\begin{pmatrix}2\\0\\4\end{pmatrix}m \mid m\in\mathbb{R}\}=\{\begin{pmatrix}1\\2/3\\0\end{pmatrix}+\begin{pmatrix}2\\8/3\\4\end{pmatrix}m \mid m\in\mathbb{R}\}$$

One.II.1.7 **(a)** The system

$$\begin{aligned}1&=1\\1+t&=3+s\\2+t&=-2+2s\end{aligned}$$

gives $s=6$ and $t=8$, so this is the solution set.

$$\{\begin{pmatrix}1\\9\\10\end{pmatrix}\}$$

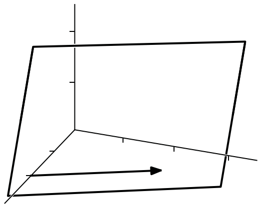
(b) This system

$$\begin{aligned}2+t&=0\\t&=s+4w\\1-t&=2s+w\end{aligned}$$

gives $t=-2$, $w=-1$, and $s=2$ so their intersection is this point.

$$\begin{pmatrix}0\\-2\\3\end{pmatrix}$$

One.II.1.8 **(a)** The vector shown



is not the result of doubling

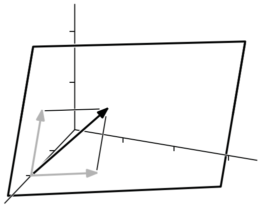
$$\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}\cdot 1$$

instead it is

$$\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}\cdot 2=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

which has a parameter twice as large.

(b) The vector



is not the result of adding

$$(\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}\cdot 1)+(\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}\cdot 1)$$

instead it is

$$\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}\cdot 1+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}\cdot 1=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

which adds the parameters.

One.II.1.9 The "if" half is straightforward. If $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$ then

$$\sqrt{(b_1-a_1)^2+(b_2-a_2)^2}=\sqrt{(d_1-c_1)^2+(d_2-c_2)^2}$$

so they have the same lengths, and the slopes are just as easy:

$$\frac{b_2-a_2}{b_1-a_1}=\frac{d_2-c_2}{d_1-a_1}$$

(if the denominators are 0 they both have undefined slopes).

For "only if", assume that the two segments have the same length and slope (the case of undefined slopes is easy; we will do the case where both segments have a slope m). Also assume, without loss of generality, that $a_1 < b_1$ and that $c_1 < d_1$. The first segment is $\overline{(a_1,a_2)(b_1,b_2)} = \{(x,y) \mid y = mx + n_1,\ x \in [a_1..b_1]\}$ (for some intercept n_1) and the second segment is $\overline{(c_1,c_2)(d_1,d_2)} = \{(x,y) \mid y = mx + n_2,\ x \in [c_1..d_1]\}$ (for some n_2). Then the lengths of those segments are

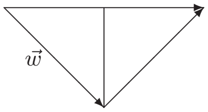
$$\sqrt{(b_1-a_1)^2+((mb_1+n_1)-(ma_1+n_1))^2}=\sqrt{(1+m^2)(b_1-a_1)^2}$$

and, similarly, $\sqrt{(1+m^2)(d_1-c_1)^2}$. Therefore, $|b_1-a_1| = |d_1-c_1|$. Thus, as we assumed that $a_1 < b_1$ and $c_1 < d_1$, we have that $b_1 - a_1 = d_1 - c_1$.

The other equality is similar.

One.II.1.10 We shall later define it to be a set with one element — an "origin".

One.II.1.11 *This is how the answer was given in the cited source.* The vector triangle is as follows, so $\vec{w} = 3\sqrt{2}$ from the north west.



One.II.1.12 Euclid no doubt is picturing a plane inside of $\mathbb{R}^3$. Observe, however, that both $\mathbb{R}^1$ and $\mathbb{R}^3$ also satisfy that definition.

Subsection One.II.2: Length and Angle Measures

One.II.2.10 **(a)** $\sqrt{3^2+1^2}=\sqrt{10}$ **(b)** $\sqrt{5}$ **(c)** $\sqrt{18}$ **(d)** 0 **(e)** $\sqrt{3}$

One.II.2.11 **(a)** $\arccos(9/\sqrt{85}) \approx 0.22$ radians **(b)** $\arccos(8/\sqrt{85}) \approx 0.52$ radians **(c)** Not defined.

One.II.2.12 We express each displacement as a vector (rounded to one decimal place because that's the accuracy of the problem's statement) and add to find the total displacement (ignoring the curvature of the earth).

$$\begin{pmatrix}0.0\\1.2\end{pmatrix}+\begin{pmatrix}3.8\\-4.8\end{pmatrix}+\begin{pmatrix}4.0\\0.1\end{pmatrix}+\begin{pmatrix}3.3\\5.6\end{pmatrix}=\begin{pmatrix}11.1\\2.1\end{pmatrix}$$

The distance is $\sqrt{11.1^2+2.1^2} \approx 11.3$.

One.II.2.13 Solve $(k)(4)+(1)(3)=0$ to get $k=-3/4$.

One.II.2.14 The set

$$\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid 1x+3y-1z=0\}$$

can also be described with parameters in this way.

$$\{\begin{pmatrix}-3\\1\\0\end{pmatrix}y+\begin{pmatrix}1\\0\\1\end{pmatrix}z \mid y,z\in\mathbb{R}\}$$

One.II.2.15 **(a)** We can use the x-axis.

$$\arccos(\frac{(1)(1)+(0)(1)}{\sqrt{1}\sqrt{2}})\approx 0.79\text{ radians}$$

(b) Again, use the x-axis.

$$\arccos(\frac{(1)(1)+(0)(1)+(0)(1)}{\sqrt{1}\sqrt{3}})\approx 0.96\text{ radians}$$

(c) The x-axis worked before and it will work again.

$$\arccos(\frac{(1)(1)+\cdots+(0)(1)}{\sqrt{1}\sqrt{n}})=\arccos(\frac{1}{\sqrt{n}})$$

(d) Using the formula from the prior item, $\lim_{n\to\infty}\arccos(1/\sqrt{n})=\pi/2$ radians.

One.II.2.16 Clearly $u_1u_1+\cdots+u_nu_n$ is zero if and only if each u_i is zero. So only $\vec{0}\in\mathbb{R}^n$ is perpendicular to itself.

One.II.2.17 Assume that $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^n$ have components $u_1,\ldots,u_n,v_1,\ldots,w_n$.

(a) Dot product is right-distributive.

$$\begin{aligned}(\vec{u}+\vec{v})\cdot\vec{w}&=[\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}+\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}]\cdot\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}\\ &=\begin{pmatrix}u_1+v_1\\ \vdots\\ u_n+v_n\end{pmatrix}\cdot\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}\\ &=(u_1+v_1)w_1+\cdots+(u_n+v_n)w_n\\ &=(u_1w_1+\cdots+u_nw_n)+(v_1w_1+\cdots+v_nw_n)\\ &=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}\end{aligned}$$

(b) Dot product is also left distributive: $\vec{w}\cdot(\vec{u}+\vec{v})=\vec{w}\cdot\vec{u}+\vec{w}\cdot\vec{v}$. The proof is just like the prior one.

(c) Dot product commutes.

$$\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}\cdot\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}=u_1v_1+\cdots+u_nv_n=v_1u_1+\cdots+v_nu_n=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\cdot\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$$

(d) Because $\vec{u}\cdot\vec{v}$ is a scalar, not a vector, the expression $(\vec{u}\cdot\vec{v})\cdot\vec{w}$ makes no sense; the dot product of a scalar and a vector is not defined.

(e) This is a vague question so it has many answers. Some are (1) $k(\vec{u}\cdot\vec{v})=(k\vec{u})\cdot\vec{v}$ and $k(\vec{u}\cdot\vec{v})=\vec{u}\cdot(k\vec{v})$, (2) $k(\vec{u}\cdot\vec{v})\neq(k\vec{u})\cdot(k\vec{v})$ (in general; an example is easy to produce), and (3) $\|k\vec{v}\,\|=|k|\|\vec{v}\,\|$ (the connection between norm and dot product is that the square of the norm is the dot product of a vector with itself).

One.II.2.18 **(a)** Verifying that $(k\vec{x})\cdot\vec{y}=k(\vec{x}\cdot\vec{y})=\vec{x}\cdot(k\vec{y})$ for $k\in\mathbb{R}$ and $\vec{x},\vec{y}\in\mathbb{R}^n$ is easy. Now, for $k\in\mathbb{R}$ and $\vec{v},\vec{w}\in\mathbb{R}^n$, if $\vec{u}=k\vec{v}$ then $\vec{u}\cdot\vec{v}=(k\vec{v})\cdot\vec{v}=k(\vec{v}\cdot\vec{v})$, which is k times a nonnegative real.

The $\vec{v}=k\vec{u}$ half is similar (actually, taking the k in this paragraph to be the reciprocal of the k above gives that we need only worry about the $k=0$ case).

(b) We first consider the $\vec{u}\cdot\vec{v}\geq 0$ case. From the Triangle Inequality we know that $\vec{u}\cdot\vec{v}=\|\vec{u}\,\|\,\|\vec{v}\,\|$ if and only if one vector is a nonnegative scalar multiple of the other. But that's all we need because the first part of this exercise shows that, in a context where the dot product of the two vectors is positive, the two statements 'one vector is a scalar multiple of the other' and 'one vector is a nonnegative scalar multiple of the other', are equivalent.

We finish by considering the $\vec{u}\cdot\vec{v}<0$ case. Because $0<|\vec{u}\cdot\vec{v}|=-(\vec{u}\cdot\vec{v})=(-\vec{u})\cdot\vec{v}$ and $\|\vec{u}\,\|\,\|\vec{v}\,\|=\|-\vec{u}\,\|\,\|\vec{v}\,\|$, we have that $0<(-\vec{u})\cdot\vec{v}=\|-\vec{u}\,\|\,\|\vec{v}\,\|$. Now the prior paragraph applies to give that one of the two vectors $-\vec{u}$ and $\vec{v}$ is a scalar multiple of the other. But that's equivalent to the assertion that one of the two vectors $\vec{u}$ and $\vec{v}$ is a scalar multiple of the other, as desired.

One.II.2.19 No. These give an example.

$$\vec{u}=\begin{pmatrix}1\\ 0\end{pmatrix}\quad\vec{v}=\begin{pmatrix}1\\ 0\end{pmatrix}\quad\vec{w}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

One.II.2.20 We prove that a vector has length zero if and only if all its components are zero.

Let $\vec{u} \in \mathbb{R}^n$ have components $u_1, \ldots, u_n$. Recall that the square of any real number is greater than or equal to zero, with equality only when that real is zero. Thus $\|\vec{u}\,\|^2 = {u_1}^2 + \cdots + {u_n}^2$ is a sum of numbers greater than or equal to zero, and so is itself greater than or equal to zero, with equality if and only if each u_i is zero. Hence $\|\vec{u}\,\| = 0$ if and only if all the components of $\vec{u}$ are zero.

One.II.2.21 We can easily check that

$$\Bigl(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\Bigr)$$

is on the line connecting the two, and is equidistant from both. The generalization is obvious.

One.II.2.22 Assume that $\vec{v} \in \mathbb{R}^n$ has components $v_1, \ldots, v_n$. If $\vec{v} \neq \vec{0}$ then we have this.

$$\begin{aligned}
&\sqrt{\left(\frac{v_1}{\sqrt{{v_1}^2 + \cdots + {v_n}^2}}\right)^2 + \cdots + \left(\frac{v_n}{\sqrt{{v_1}^2 + \cdots + {v_n}^2}}\right)^2} \\
&\qquad = \sqrt{\left(\frac{{v_1}^2}{{v_1}^2 + \cdots + {v_n}^2}\right) + \cdots + \left(\frac{{v_n}^2}{{v_1}^2 + \cdots + {v_n}^2}\right)} \\
&\qquad = 1
\end{aligned}$$

If $\vec{v} = \vec{0}$ then $\vec{v}/\|\vec{v}\,\|$ is not defined.

One.II.2.23 For the first question, assume that $\vec{v} \in \mathbb{R}^n$ and $r \geq 0$, take the root, and factor.

$$\|r\vec{v}\,\| = \sqrt{(rv_1)^2 + \cdots + (rv_n)^2} = \sqrt{r^2({v_1}^2 + \cdots + {v_n}^2} = r\|\vec{v}\,\|$$

For the second question, the result is r times as long, but it points in the opposite direction in that $r\vec{v} + (-r)\vec{v} = \vec{0}$.

One.II.2.24 Assume that $\vec{u}, \vec{v} \in \mathbb{R}^n$ both have length 1. Apply Cauchy-Schwartz: $|\vec{u}\bullet\vec{v}| \leq \|\vec{u}\,\|\,\|\vec{v}\,\| = 1$.

To see that 'less than' can happen, in $\mathbb{R}^2$ take

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and note that $\vec{u} \bullet \vec{v} = 0$. For 'equal to', note that $\vec{u} \bullet \vec{u} = 1$.

One.II.2.25 Write

$$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and then this computation works.

$$\begin{aligned}
\|\vec{u} + \vec{v}\,\|^2 + \|\vec{u} - \vec{v}\,\|^2 &= (u_1 + v_1)^2 + \cdots + (u_n + v_n)^2 \\
&\quad + (u_1 - v_1)^2 + \cdots + (u_n - v_n)^2 \\
&= {u_1}^2 + 2u_1v_1 + {v_1}^2 + \cdots + {u_n}^2 + 2u_nv_n + {v_n}^2 \\
&\quad + {u_1}^2 - 2u_1v_1 + {v_1}^2 + \cdots + {u_n}^2 - 2u_nv_n + {v_n}^2 \\
&= 2({u_1}^2 + \cdots + {u_n}^2) + 2({v_1}^2 + \cdots + {v_n}^2) \\
&= 2\|\vec{u}\,\|^2 + 2\|\vec{v}\,\|^2
\end{aligned}$$

One.II.2.26 We will prove this demonstrating that the contrapositive statement holds: if $\vec{x} \neq \vec{0}$ then there is a $\vec{y}$ with $\vec{x} \bullet \vec{y} \neq 0$.

Assume that $\vec{x} \in \mathbb{R}^n$. If $\vec{x} \neq \vec{0}$ then it has a nonzero component, say the i-th one x_i. But the vector $\vec{y} \in \mathbb{R}^n$ that is all zeroes except for a one in component i gives $\vec{x} \bullet \vec{y} = x_i$. (A slicker proof just considers $\vec{x} \bullet \vec{x}$.)

One.II.2.27 Yes; we can prove this by induction.

Assume that the vectors are in some $\mathbb{R}^k$. Clearly the statement applies to one vector. The Triangle Inequality is this statement applied to two vectors. For an inductive step assume the statement is true for n or fewer vectors. Then this

$$\|\vec{u}_1 + \cdots + \vec{u}_n + \vec{u}_{n+1}\| \leq \|\vec{u}_1 + \cdots + \vec{u}_n\| + \|\vec{u}_{n+1}\|$$

follows by the Triangle Inequality for two vectors. Now the inductive hypothesis, applied to the first summand on the right, gives that as less than or equal to $\|\vec{u}_1\| + \cdots + \|\vec{u}_n\| + \|\vec{u}_{n+1}\|$.

One.II.2.28 By definition

$$\frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\,\|\,\|\vec{v}\,\|}=\cos\theta$$

where θ is the angle between the vectors. Thus the ratio is $|\cos\theta|$.

One.II.2.29 So that the statement 'vectors are orthogonal iff their dot product is zero' has no exceptions.

One.II.2.30 The angle between (a) and (b) is found (for $a, b \neq 0$) with

$$\arccos(\frac{ab}{\sqrt{a^2}\sqrt{b^2}}).$$

If a or b is zero then the angle is $\pi/2$ radians. Otherwise, if a and b are of opposite signs then the angle is π radians, else the angle is zero radians.

One.II.2.31 The angle between $\vec{u}$ and $\vec{v}$ is acute if $\vec{u}\bullet\vec{v} > 0$, is right if $\vec{u}\bullet\vec{v} = 0$, and is obtuse if $\vec{u}\bullet\vec{v} < 0$. That's because, in the formula for the angle, the denominator is never negative.

One.II.2.32 Suppose that $\vec{u}, \vec{v} \in \mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are perpendicular then

$$\|\vec{u}+\vec{v}\,\|^2=(\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v})=\vec{u}\bullet\vec{u}+2\,\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{v}=\vec{u}\bullet\vec{u}+\vec{v}\bullet\vec{v}=\|\vec{u}\,\|^2+\|\vec{v}\,\|^2$$

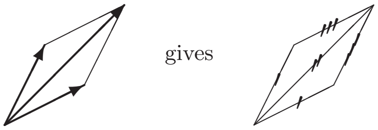
(the third equality holds because $\vec{u}\bullet\vec{v} = 0$).

One.II.2.33 Where $\vec{u}, \vec{v} \in \mathbb{R}^n$, the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular if and only if $0 = (\vec{u}+\vec{v})\bullet(\vec{u}-\vec{v}) = \vec{u}\bullet\vec{u}-\vec{v}\bullet\vec{v}$, which shows that those two are perpendicular if and only if $\vec{u}\bullet\vec{u} = \vec{v}\bullet\vec{v}$. That holds if and only if $\|\vec{u}\,\| = \|\vec{v}\,\|$.

One.II.2.34 Suppose $\vec{u} \in \mathbb{R}^n$ is perpendicular to both $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$. Then, for any $k, m \in \mathbb{R}$ we have this.

$$\vec{u}\bullet(k\vec{v}+m\vec{w})=k(\vec{u}\bullet\vec{v})+m(\vec{u}\bullet\vec{w})=k(0)+m(0)=0$$

One.II.2.35 We will show something more general: if $\|\vec{z}_1\| = \|\vec{z}_2\|$ for $\vec{z}_1, \vec{z}_2 \in \mathbb{R}^n$, then $\vec{z}_1+\vec{z}_2$ bisects the angle between $\vec{z}_1$ and $\vec{z}_2$



(we ignore the case where $\vec{z}_1$ and $\vec{z}_2$ are the zero vector).

The $\vec{z}_1+\vec{z}_2=\vec{0}$ case is easy. For the rest, by the definition of angle, we will be done if we show this.

$$\frac{\vec{z}_1\bullet(\vec{z}_1+\vec{z}_2)}{\|\vec{z}_1\|\,\|\vec{z}_1+\vec{z}_2\|}=\frac{\vec{z}_2\bullet(\vec{z}_1+\vec{z}_2)}{\|\vec{z}_2\|\,\|\vec{z}_1+\vec{z}_2\|}$$

But distributing inside each expression gives

$$\frac{\vec{z}_1\bullet\vec{z}_1+\vec{z}_1\bullet\vec{z}_2}{\|\vec{z}_1\|\,\|\vec{z}_1+\vec{z}_2\|}\qquad\frac{\vec{z}_2\bullet\vec{z}_1+\vec{z}_2\bullet\vec{z}_2}{\|\vec{z}_2\|\,\|\vec{z}_1+\vec{z}_2\|}$$

and $\vec{z}_1\bullet\vec{z}_1 = \|\vec{z}_1\|^2 = \|\vec{z}_2\|^2 = \vec{z}_2\bullet\vec{z}_2$, so the two are equal.

One.II.2.36 We can show the two statements together. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, write

$$\vec{u}=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}\qquad\vec{v}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$$

and calculate.

$$\cos\theta=\frac{ku_1v_1+\cdots+ku_nv_n}{\sqrt{(ku_1)^2+\cdots+(ku_n)^2}\sqrt{b_1{}^2+\cdots+b_n{}^2}}=\frac{k}{|k|}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\,\|\,\|\vec{v}\,\|}=\pm\frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\,\|\,\|\vec{v}\,\|}$$

One.II.2.37 Let

$$\vec{u}=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix},\quad\vec{v}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\quad\vec{w}=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}$$

and then

$$\begin{aligned}
\vec{u}\cdot(k\vec{v}+m\vec{w}) &= \begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}\cdot\Bigl(\begin{pmatrix}kv_1\\ \vdots\\ kv_n\end{pmatrix}+\begin{pmatrix}mw_1\\ \vdots\\ mw_n\end{pmatrix}\Bigr)\\
&= \begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}\cdot\begin{pmatrix}kv_1+mw_1\\ \vdots\\ kv_n+mw_n\end{pmatrix}\\
&= u_1(kv_1+mw_1)+\cdots+u_n(kv_n+mw_n)\\
&= ku_1v_1+mu_1w_1+\cdots+ku_nv_n+mu_nw_n\\
&= (ku_1v_1+\cdots+ku_nv_n)+(mu_1w_1+\cdots+mu_nw_n)\\
&= k(\vec{u}\cdot\vec{v})+m(\vec{u}\cdot\vec{w})
\end{aligned}$$

as required.

One.II.2.38 For $x, y \in \mathbb{R}^+$, set

$$\vec{u}=\begin{pmatrix}\sqrt{x}\\ \sqrt{y}\end{pmatrix} \qquad \vec{v}=\begin{pmatrix}\sqrt{y}\\ \sqrt{x}\end{pmatrix}$$

so that the Cauchy-Schwartz inequality asserts that (after squaring)

$$\begin{aligned}
(\sqrt{x}\sqrt{y}+\sqrt{y}\sqrt{x})^2 &\leq (\sqrt{x}\sqrt{x}+\sqrt{y}\sqrt{y})(\sqrt{y}\sqrt{y}+\sqrt{x}\sqrt{x})\\
(2\sqrt{x}\sqrt{y})^2 &\leq (x+y)^2\\
\sqrt{xy} &\leq \frac{x+y}{2}
\end{aligned}$$

as desired.

One.II.2.39 *This is how the answer was given in the cited source.* The actual velocity $\vec{v}$ of the wind is the sum of the ship's velocity and the apparent velocity of the wind. Without loss of generality we may assume $\vec{a}$ and $\vec{b}$ to be unit vectors, and may write

$$\vec{v}=\vec{v}_1+s\vec{a}=\vec{v}_2+t\vec{b}$$

where s and t are undetermined scalars. Take the dot product first by $\vec{a}$ and then by $\vec{b}$ to obtain

$$\begin{aligned}
s-t\vec{a}\cdot\vec{b} &= \vec{a}\cdot(\vec{v}_2-\vec{v}_1)\\
s\vec{a}\cdot\vec{b}-t &= \vec{b}\cdot(\vec{v}_2-\vec{v}_1)
\end{aligned}$$

Multiply the second by $\vec{a}\cdot\vec{b}$, subtract the result from the first, and find

$$s=\frac{[\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}]\cdot(\vec{v}_2-\vec{v}_1)}{1-(\vec{a}\cdot\vec{b})^2}.$$

Substituting in the original displayed equation, we get

$$\vec{v}=\vec{v}_1+\frac{[\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}]\cdot(\vec{v}_2-\vec{v}_1)\vec{a}}{1-(\vec{a}\cdot\vec{b})^2}.$$

One.II.2.40 We use induction on n.

In the $n=1$ base case the identity reduces to

$$(a_1b_1)^2=({a_1}^2)({b_1}^2)-0$$

and clearly holds.

For the inductive step assume that the formula holds for the $0, \ldots, n$ cases. We will show that it then holds in the $n+1$ case. Start with the right-hand side

$$
\begin{aligned}
\Bigl(\sum_{1\le j\le n+1} {a_j}^2\Bigr)\Bigl(\sum_{1\le j\le n+1} {b_j}^2\Bigr) - \sum_{1\le k<j\le n+1}\bigl(a_kb_j-a_jb_k\bigr)^2
&= \Bigl[\Bigl(\sum_{1\le j\le n} {a_j}^2\Bigr)+{a_{n+1}}^2\Bigr]\Bigl[\Bigl(\sum_{1\le j\le n} {b_j}^2\Bigr)+{b_{n+1}}^2\Bigr] \\
&\quad -\Bigl[\sum_{1\le k<j\le n}\bigl(a_kb_j-a_jb_k\bigr)^2+\sum_{1\le k\le n}\bigl(a_kb_{n+1}-a_{n+1}b_k\bigr)^2\Bigr] \\
&= \Bigl(\sum_{1\le j\le n} {a_j}^2\Bigr)\Bigl(\sum_{1\le j\le n} {b_j}^2\Bigr)+\sum_{1\le j\le n}{b_j}^2{a_{n+1}}^2+\sum_{1\le j\le n}{a_j}^2{b_{n+1}}^2+{a_{n+1}}^2{b_{n+1}}^2 \\
&\quad -\Bigl[\sum_{1\le k<j\le n}\bigl(a_kb_j-a_jb_k\bigr)^2+\sum_{1\le k\le n}\bigl(a_kb_{n+1}-a_{n+1}b_k\bigr)^2\Bigr] \\
&= \Bigl(\sum_{1\le j\le n} {a_j}^2\Bigr)\Bigl(\sum_{1\le j\le n} {b_j}^2\Bigr)-\sum_{1\le k<j\le n}\bigl(a_kb_j-a_jb_k\bigr)^2 \\
&\quad +\sum_{1\le j\le n}{b_j}^2{a_{n+1}}^2+\sum_{1\le j\le n}{a_j}^2{b_{n+1}}^2+{a_{n+1}}^2{b_{n+1}}^2 \\
&\quad -\sum_{1\le k\le n}\bigl(a_kb_{n+1}-a_{n+1}b_k\bigr)^2
\end{aligned}
$$

and apply the inductive hypothesis

$$
\begin{aligned}
&= \Bigl(\sum_{1\le j\le n} a_jb_j\Bigr)^2+\sum_{1\le j\le n}{b_j}^2{a_{n+1}}^2+\sum_{1\le j\le n}{a_j}^2{b_{n+1}}^2+{a_{n+1}}^2{b_{n+1}}^2 \\
&\quad -\Bigl[\sum_{1\le k\le n}{a_k}^2{b_{n+1}}^2-2\sum_{1\le k\le n}a_kb_{n+1}a_{n+1}b_k+\sum_{1\le k\le n}{a_{n+1}}^2{b_k}^2\Bigr] \\
&= \Bigl(\sum_{1\le j\le n} a_jb_j\Bigr)^2+2\Bigl(\sum_{1\le k\le n}a_kb_{n+1}a_{n+1}b_k\Bigr)+{a_{n+1}}^2{b_{n+1}}^2 \\
&= \Bigl[\Bigl(\sum_{1\le j\le n} a_jb_j\Bigr)+a_{n+1}b_{n+1}\Bigr]^2
\end{aligned}
$$

to derive the left-hand side.

Subsection One.III.1: Gauss-Jordan Reduction

One.III.1.7 These answers show only the Gauss-Jordan reduction. With it, describing the solution set is easy.

(a) $\left(\begin{array}{cc|c}1&1&2\\1&-1&0\end{array}\right)\xrightarrow{-\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&-2&-2\end{array}\right)\xrightarrow{-(1/2)\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&1&1\end{array}\right)\xrightarrow{-\rho_2+\rho_1}\left(\begin{array}{cc|c}1&0&1\\0&1&1\end{array}\right)$

(b) $\left(\begin{array}{ccc|c}1&0&-1&4\\2&2&0&1\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&0&-1&4\\0&2&2&-7\end{array}\right)\xrightarrow{(1/2)\rho_2}\left(\begin{array}{ccc|c}1&0&-1&4\\0&1&1&-7/2\end{array}\right)$

(c)

$$\left(\begin{array}{cc|c}3&-2&1\\6&1&1/2\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{cc|c}3&-2&1\\0&5&-3/2\end{array}\right)\xrightarrow[(1/5)\rho_2]{(1/3)\rho_1}\left(\begin{array}{cc|c}1&-2/3&1/3\\0&1&-3/10\end{array}\right)\xrightarrow{(2/3)\rho_2+\rho_1}\left(\begin{array}{cc|c}1&0&2/15\\0&1&-3/10\end{array}\right)$$

(d) A row swap here makes the arithmetic easier.

$$
\begin{aligned}
\left(\begin{array}{ccc|c}2&-1&0&-1\\1&3&-1&5\\0&1&2&5\end{array}\right)&\xrightarrow{-(1/2)\rho_1+\rho_2}\left(\begin{array}{ccc|c}2&-1&0&-1\\0&7/2&-1&11/2\\0&1&2&5\end{array}\right)\xrightarrow{\rho_2\leftrightarrow\rho_3}\left(\begin{array}{ccc|c}2&-1&0&-1\\0&1&2&5\\0&7/2&-1&11/2\end{array}\right) \\
&\xrightarrow{-(7/2)\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&-1&0&-1\\0&1&2&5\\0&0&-8&-12\end{array}\right)\xrightarrow[-(1/8)\rho_2]{(1/2)\rho_1}\left(\begin{array}{ccc|c}1&-1/2&0&-1/2\\0&1&2&5\\0&0&1&3/2\end{array}\right) \\
&\xrightarrow{-2\rho_3+\rho_2}\left(\begin{array}{ccc|c}1&-1/2&0&-1/2\\0&1&0&2\\0&0&1&3/2\end{array}\right)\xrightarrow{(1/2)\rho_2+\rho_1}\left(\begin{array}{ccc|c}1&0&0&1/2\\0&1&0&2\\0&0&1&3/2\end{array}\right)
\end{aligned}
$$

One.III.1.8 Use Gauss-Jordan reduction.

(a) $\xrightarrow{-(1/2)\rho_1+\rho_2}\begin{pmatrix}2&1\\0&5/2\end{pmatrix}\xrightarrow[(2/5)\rho_2]{(1/2)\rho_1}\begin{pmatrix}1&1/2\\0&1\end{pmatrix}\xrightarrow{-(1/2)\rho_2+\rho_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(b) $\xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2}\begin{pmatrix}1&3&1\\0&-6&2\\0&0&-2\end{pmatrix}\xrightarrow[-(1/2)\rho_3]{-(1/6)\rho_2}\begin{pmatrix}1&3&1\\0&1&-1/3\\0&0&1\end{pmatrix}\xrightarrow[-\rho_3+\rho_1]{(1/3)\rho_3+\rho_2}\begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{-3\rho_2+\rho_1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

(c)

$$\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\begin{pmatrix}1&0&3&1&2\\0&4&-1&0&3\\0&4&-1&-2&-4\end{pmatrix}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&0&3&1&2\\0&4&-1&0&3\\0&0&0&-2&-7\end{pmatrix}$$
$$\xrightarrow[-(1/2)\rho_3]{(1/4)\rho_2}\begin{pmatrix}1&0&3&1&2\\0&1&-1/4&0&3/4\\0&0&0&1&7/2\end{pmatrix}\xrightarrow{-\rho_3+\rho_1}\begin{pmatrix}1&0&3&0&-3/2\\0&1&-1/4&0&3/4\\0&0&0&1&7/2\end{pmatrix}$$

(d)

$$\xrightarrow{\rho_1\leftrightarrow\rho_3}\begin{pmatrix}1&5&1&5\\0&0&5&6\\0&1&3&2\end{pmatrix}\xrightarrow{\rho_2\leftrightarrow\rho_3}\begin{pmatrix}1&5&1&5\\0&1&3&2\\0&0&5&6\end{pmatrix}\xrightarrow{(1/5)\rho_3}\begin{pmatrix}1&5&1&5\\0&1&3&2\\0&0&1&6/5\end{pmatrix}$$
$$\xrightarrow[-\rho_3+\rho_1]{-3\rho_3+\rho_2}\begin{pmatrix}1&5&0&19/5\\0&1&0&-8/5\\0&0&1&6/5\end{pmatrix}\xrightarrow{-5\rho_2+\rho_1}\begin{pmatrix}1&0&0&59/5\\0&1&0&-8/5\\0&0&1&6/5\end{pmatrix}$$

One.III.1.9 For the "Gauss" halves, see the answers to Exercise 19.

(a) The "Jordan" half goes this way.

$$\xrightarrow[-(1/3)\rho_2]{(1/2)\rho_1}\left(\begin{array}{ccc|c}1&1/2&-1/2&1/2\\0&1&-2/3&-1/3\end{array}\right)\xrightarrow{-(1/2)\rho_2+\rho_1}\left(\begin{array}{ccc|c}1&0&-1/6&2/3\\0&1&-2/3&-1/3\end{array}\right)$$

The solution set is this

$$\{\begin{pmatrix}2/3\\-1/3\\0\end{pmatrix}+\begin{pmatrix}1/6\\2/3\\1\end{pmatrix}z\mid z\in\mathbb{R}\}$$

(b) The second half is

$$\xrightarrow{\rho_3+\rho_2}\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&0&3\\0&0&0&1&0\end{array}\right)$$

so the solution is this.

$$\{\begin{pmatrix}1\\3\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}z\mid z\in\mathbb{R}\}$$

(c) This Jordan half

$$\xrightarrow{\rho_2+\rho_1}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)$$

gives

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w\mid z,w\in\mathbb{R}\}$$

(of course, the zero vector could be omitted from the description).

(d) The "Jordan" half

$$\xrightarrow{-(1/7)\rho_2}\left(\begin{array}{ccccc|c}1&2&3&1&-1&1\\0&1&8/7&2/7&-4/7&0\end{array}\right)\xrightarrow{-2\rho_2+\rho_1}\left(\begin{array}{ccccc|c}1&0&5/7&3/7&1/7&1\\0&1&8/7&2/7&-4/7&0\end{array}\right)$$

ends with this solution set.

$$\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-5/7\\-8/7\\1\\0\\0\end{pmatrix}c+\begin{pmatrix}-3/7\\-2/7\\0\\1\\0\end{pmatrix}d+\begin{pmatrix}-1/7\\4/7\\0\\0\\1\end{pmatrix}e\mid c,d,e\in\mathbb{R}\}$$

One.III.1.10 Routine Gauss' method gives one:

$$\xrightarrow[-(1/2)\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}2&1&1&3\\0&1&-2&-7\\0&9/2&1/2&7/2\end{pmatrix}\xrightarrow{-(9/2)\rho_2+\rho_3}\begin{pmatrix}2&1&1&3\\0&1&-2&-7\\0&0&19/2&35\end{pmatrix}$$

and any cosmetic change, like multiplying the bottom row by 2,

$$\begin{pmatrix}2&1&1&3\\0&1&-2&-7\\0&0&19&70\end{pmatrix}$$

gives another.

One.III.1.11 In the cases listed below, we take $a, b \in \mathbb{R}$. Thus, some canonical forms listed below actually include infinitely many cases. In particular, they includes the cases $a = 0$ and $b = 0$.

(a) $\begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&a\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}$

(b) $\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&a&b\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&a\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&a\\0&1&b\end{pmatrix}, \begin{pmatrix}1&a&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}$

(c) $\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&a\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$

(d) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&a&b\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&a\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&a\\0&1&b\\0&0&0\end{pmatrix}, \begin{pmatrix}1&a&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

One.III.1.12 A nonsingular homogeneous linear system has a unique solution. So a nonsingular matrix must reduce to a (square) matrix that is all 0's except for 1's down the upper-left to lower-right diagonal, e.g.,

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \text{etc.}$$

One.III.1.13 **(a)** The $\rho_i \leftrightarrow \rho_i$ operation does not change A.

(b) For instance,

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\xrightarrow{-\rho_1+\rho_1}\begin{pmatrix}0&0\\3&4\end{pmatrix}\xrightarrow{\rho_1+\rho_1}\begin{pmatrix}0&0\\3&4\end{pmatrix}$$

leaves the matrix changed.

(c) If $i \neq j$ then

$$\begin{pmatrix}\vdots\\a_{i,1}&\cdots&a_{i,n}\\\vdots\\a_{j,1}&\cdots&a_{j,n}\\\vdots\end{pmatrix}\xrightarrow{k\rho_i+\rho_j}\begin{pmatrix}\vdots\\a_{i,1}&\cdots&a_{i,n}\\\vdots\\ka_{i,1}+a_{j,1}&\cdots&ka_{i,n}+a_{j,n}\\\vdots\end{pmatrix}$$

$$\xrightarrow{-k\rho_i+\rho_j}\begin{pmatrix}\vdots\\a_{i,1}&\cdots&a_{i,n}\\\vdots\\-ka_{i,1}+ka_{i,1}+a_{j,1}&\cdots&-ka_{i,n}+ka_{i,n}+a_{j,n}\\\vdots\end{pmatrix}$$

does indeed give A back. (Of course, if $i = j$ then the third matrix would have entries of the form $-k(ka_{i,j}+a_{i,j})+ka_{i,j}+a_{i,j}$.)

Subsection One.III.2: Row Equivalence

One.III.2.11 Bring each to reduced echelon form and compare.

(a) The first gives

$$\xrightarrow{-4\rho_1+\rho_2}\begin{pmatrix}1&2\\0&0\end{pmatrix}$$

while the second gives

$$\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}1&2\\0&1\end{pmatrix}\xrightarrow{-2\rho_2+\rho_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

The two reduced echelon form matrices are not identical, and so the original matrices are not row equivalent.

(b) The first is this.

$$\xrightarrow[-5\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}1&0&2\\0&-1&-5\\0&-1&-5\end{pmatrix}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&0&2\\0&-1&-5\\0&0&0\end{pmatrix}\xrightarrow{-\rho_2}\begin{pmatrix}1&0&2\\0&1&5\\0&0&0\end{pmatrix}$$

The second is this.

$$\xrightarrow{-2\rho_1+\rho_3}\begin{pmatrix}1&0&2\\0&2&10\\0&0&0\end{pmatrix}\xrightarrow{(1/2)\rho_2}\begin{pmatrix}1&0&2\\0&1&5\\0&0&0\end{pmatrix}$$

These two are row equivalent.

(c) These two are not row equivalent because they have different sizes.

(d) The first,

$$\xrightarrow{\rho_1+\rho_2}\begin{pmatrix}1&1&1\\0&3&3\end{pmatrix}\xrightarrow{(1/3)\rho_2}\begin{pmatrix}1&1&1\\0&1&1\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}$$

and the second.

$$\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}2&2&5\\0&3&-1\end{pmatrix}\xrightarrow[(1/3)\rho_2]{(1/2)\rho_1}\begin{pmatrix}1&1&5/2\\0&1&-1/3\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&0&17/6\\0&1&-1/3\end{pmatrix}$$

These are not row equivalent.

(e) Here the first is

$$\xrightarrow{(1/3)\rho_2}\begin{pmatrix}1&1&1\\0&0&1\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}$$

while this is the second.

$$\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}1&-1&1\\0&1&2\end{pmatrix}\xrightarrow{\rho_2+\rho_1}\begin{pmatrix}1&0&3\\0&1&2\end{pmatrix}$$

These are not row equivalent.

One.III.2.12 First, the only matrix row equivalent to the matrix of all 0's is itself (since row operations have no effect).

Second, the matrices that reduce to

$$\begin{pmatrix}1&a\\0&0\end{pmatrix}$$

have the form

$$\begin{pmatrix}b&ba\\c&ca\end{pmatrix}$$

(where $a,b,c\in\mathbb{R}$, and b and c are not both zero).

Next, the matrices that reduce to

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

have the form

$$\begin{pmatrix}0&a\\0&b\end{pmatrix}$$

(where $a,b\in\mathbb{R}$, and not both are zero).

Finally, the matrices that reduce to

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

are the nonsingular matrices. That's because a linear system for which this is the matrix of coefficients will have a unique solution, and that is the definition of nonsingular. (Another way to say the same thing is to say that they fall into none of the above classes.)

One.III.2.13 **(a)** They have the form

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

where $a, b \in \mathbb{R}$.

(b) They have this form (for $a, b \in \mathbb{R}$).

$$\begin{pmatrix} 1a & 2a \\ 1b & 2b \end{pmatrix}$$

(c) They have the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(for $a, b, c, d \in \mathbb{R}$) where $ad - bc \neq 0$. (This is the formula that determines when a 2×2 matrix is nonsingular.)

One.III.2.14 Infinitely many. For instance, in

$$\begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}$$

each $k \in \mathbb{R}$ gives a different class.

One.III.2.15 No. Row operations do not change the size of a matrix.

One.III.2.16 **(a)** A row operation on a zero matrix has no effect. Thus each zero matrix is alone in its row equivalence class.

(b) No. Any nonzero entry can be rescaled.

One.III.2.17 Here are two.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One.III.2.18 Any two $n\times n$ nonsingular matrices have the same reduced echelon form, namely the matrix with all 0's except for 1's down the diagonal.

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

Two same-sized singular matrices need not be row equivalent. For example, these two 2×2 singular matrices are not row equivalent.

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

One.III.2.19 Since there is one and only one reduced echelon form matrix in each class, we can just list the possible reduced echelon form matrices.

For that list, see the answer for Exercise 11.

One.III.2.20 **(a)** If there is a linear relationship where c_0 is not zero then we can subtract $c_0\vec{\beta}_0$ from both sides and divide by $-c_0$ to get $\vec{\beta}_0$ as a linear combination of the others. (Remark: if there are no other vectors in the set — if the relationship is, say, $\vec{0} = 3 \cdot \vec{0}$ — then the statement is still true because the zero vector is by definition the sum of the empty set of vectors.)

Conversely, if $\vec{\beta}_0$ is a combination of the others $\vec{\beta}_0 = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ then subtracting $\vec{\beta}_0$ from both sides gives a relationship where at least one of the coefficients is nonzero; namely, the -1 in front of $\vec{\beta}_0$.

(b) The first row is not a linear combination of the others for the reason given in the proof: in the equation of components from the column containing the leading entry of the first row, the only nonzero entry is the leading entry from the first row, so its coefficient must be zero. Thus, from the prior part of this exercise, the first row is in no linear relationship with the other rows.

Thus, when considering whether the second row can be in a linear relationship with the other rows, we can leave the first row out. But now the argument just applied to the first row will apply to the second row. (That is, we are arguing here by induction.)

One.III.2.21 **(a)** In the equation

$$\rho_i = c_1\rho_1 + c_2\rho_2 + \ldots + c_{i-1}\rho_{i-1} + c_{i+1}\rho_{i+1} + \ldots + c_m\rho_m$$

we already know that $c_1 = 0$. Let ℓ_2 be the column number of the leading entry of the second row. Consider the prior equation on entries in that column.

$$\rho_{i,\ell_1} = c_2\rho_{2,\ell_2} + \ldots + c_{i-1}\rho_{i-1,\ell_2} + c_{i+1}\rho_{i+1,\ell_2} + \ldots + c_m\rho_{m,\ell_2}$$

Because ℓ_2 is the column of the leading entry in the second row, $\rho_{i,\ell_2} = 0$ for $i > 2$. Thus the equation reduces to

$$0 = c_2\rho_{2,\ell_2} + 0 + \ldots + 0$$

and since ρ_{2,ℓ_2} is not 0 we have that $c_2 = 0$.

(b) In the equation

$$\rho_i = c_1\rho_1 + c_2\rho_2 + \ldots + c_{i-1}\rho_{i-1} + c_{i+1}\rho_{i+1} + \ldots + c_m\rho_m$$

we already know that $0 = c_1 = c_2 = \cdots = c_n$. Let ℓ_{n+1} be the column number of the leading entry of row $n+1$. Consider the above equation on entries in that column.

$$\rho_{i,\ell_{n+1}} = c_{n+1}\rho_{n+1,\ell_{n+1}} + \ldots + c_{i-1}\rho_{i-1,\ell_{n+1}} + c_{i+1}\rho_{i+1,\ell_{n+1}} + \cdots + c_m\rho_{m,\ell_{n+1}}$$

Because ℓ_{n+1} is the column of the leading entry in the row $n+1$, we have that $\rho_{j,\ell_{n+1}} = 0$ for $j > n+1$. Thus the equation reduces to

$$0 = c_{n+1}\rho_{n+1,\ell_{n+1}} + 0 + \ldots + 0$$

and since $\rho_{n+1,\ell_{n+1}}$ is not 0 we have that $c_{n+1} = 0$.

(c) From the prior item in this exercise we know that in the equation

$$\rho_i = c_1\rho_1 + c_2\rho_2 + \ldots + c_{i-1}\rho_{i-1} + c_{i+1}\rho_{i+1} + \ldots + c_m\rho_m$$

we already know that $0 = c_1 = c_2 = \cdots = c_{i-1}$.

Let ℓ_i be the column number of the leading entry of row i. Rewrite the above equation on entries in that column.

$$\rho_{i,\ell_i} = c_{i+1}\rho_{i+1,\ell_i} + \cdots + c_m\rho_{m,\ell_i}$$

Because ℓ_i is the column of the leading entry in the row i, we have that $\rho_{j,\ell_i} = 0$ for $j > i$. That makes the right side of the equation sum to 0, but the left side is not 0 since it is the leading entry of the row. That's the contradiction.

One.III.2.22 **(a)** The inductive step is to show that if the statement holds on rows 1 through r then it also holds on row $r+1$. That is, we assume that $\ell_1 = k_1$, and $\ell_2 = k_2$, ..., and $\ell_r = k_r$, and we will show that $\ell_{r+1} = k_{r+1}$ also holds (for r in $1\,..\,m-1$).

(b) Lemma 2.3 gives the relationship $\beta_{r+1} = s_{r+1,1}\delta_1 + s_{r+2,2}\delta_2 + \cdots + s_{r+1,m}\delta_m$ between rows.

Inside of those row vectors, consider the relationship between the entries in the column $\ell_1 = k_1$. Because by the induction hypothesis this is a row greater than the first $r+1 > 1$, the row β_{r+1} has a zero in entry ℓ_1 (the matrix B is in echelon form). But the row δ_1 has a nonzero entry in column k_1; by definition of k_1 it is the leading entry in the first row of D. Thus, in that column, the above relationship among rows resolves to this equation among numbers: $0 = s_{r+1,1} \cdot d_{1,k_1}$, with $d_{1,k_1} \neq 0$. Therefore $s_{r+1,1} = 0$.

With $s_{r+1,1} = 0$, a similar argument shows that $s_{r+1,2} = 0$. With those two, another turn gives that $s_{r+1,3} = 0$. That is, inside of the larger induction argument used to prove the entire lemma, here is an subargument by induction that shows $s_{r+1,j} = 0$ for all j in $1\,..\,r$. (We won't write out the details since it is just like the induction done in Exercise 21.)

(c) Note that the prior item of this exercise shows that the relationship between rows $\beta_{r+1} = s_{r+1,1}\delta_1 + s_{r+2,2}\delta_2 + \cdots + s_{r+1,m}\delta_m$ reduces to $\beta_{r+1} = s_{r+1,r+1}\delta_{r+1} + \cdots + s_{r+1,m}\delta_m$. Consider the column ℓ_{r+1} entries in this equation. By definition of k_{r+1} as the column number of the leading entry of δ_{r+1}, the entries in this column of the other rows $\delta_{r+2}\ \ \delta_m$ are zeros. Now if $\ell_{r+1} < k_{r+1}$ then the equation of entries from column ℓ_{k+1} would be $b_{r+1,\ell_{r+1}} = s_{r+1,1} \cdot 0 + \cdots + s_{r+1,m} \cdot 0$, which is impossible as $b_{r+1,\ell_{r+1}}$ isn't zero as it leads its row.

A symmetric argument shows that $k_{r+1} < \ell_{r+1}$ also is impossible.

One.III.2.23 The zero rows could have nonzero coefficients, and so the statement would not be true.

One.III.2.24 We know that $4s + c + 10d = 8.45$ and that $3s + c + 7d = 6.30$, and we'd like to know what $s + c + d$ is. Fortunately, $s + c + d$ is a linear combination of $4s + c + 10d$ and $3s + c + 7d$. Calling the unknown price p, we have this reduction.

$$\left(\begin{array}{ccc|c} 4 & 1 & 10 & 8.45 \\ 3 & 1 & 7 & 6.30 \\ 1 & 1 & 1 & p \end{array}\right) \xrightarrow[-(1/4)\rho_1+\rho_3]{-(3/4)\rho_1+\rho_2} \left(\begin{array}{ccc|c} 4 & 1 & 10 & 8.45 \\ 0 & 1/4 & -1/2 & -0.0375 \\ 0 & 3/4 & -3/2 & p-2.1125 \end{array}\right) \xrightarrow{-3\rho_2+\rho_3} \left(\begin{array}{ccc|c} 4 & 1 & 10 & 8.45 \\ 0 & 1/4 & -1/2 & -0.0375 \\ 0 & 0 & 0 & p-2.00 \end{array}\right)$$

The price paid is \$2.00.

One.III.2.25 If multiplication of a row by zero were allowed then Lemma 2.6 would not hold. That is, where

$$\begin{pmatrix}1&3\\2&1\end{pmatrix}\xrightarrow{0\rho_2}\begin{pmatrix}1&3\\0&0\end{pmatrix}$$

all the rows of the second matrix can be expressed as linear combinations of the rows of the first, but the converse does not hold. The second row of the first matrix is not a linear combination of the rows of the second matrix.

One.III.2.26 (1) An easy answer is this:

$$0=3.$$

For a less wise-guy-ish answer, solve the system:

$$\left(\begin{array}{cc|c}3&-1&8\\2&1&3\end{array}\right)\xrightarrow{-(2/3)\rho_1+\rho_2}\left(\begin{array}{cc|c}3&-1&8\\0&5/3&-7/3\end{array}\right)$$

gives $y=-7/5$ and $x=11/5$. Now any equation not satisfied by $(-7/5,11/5)$ will do, e.g., $5x+5y=3$.

(2) Every equation can be derived from an inconsistent system. For instance, here is how to derive "$3x+2y=4$" from "$0=5$". First,

$$0=5\xrightarrow{(3/5)\rho_1}0=3\xrightarrow{x\rho_1}0=3x$$

(validity of the $x=0$ case is separate but clear). Similarly, $0=2y$. Ditto for $0=4$. But now, $0+0=0$ gives $3x+2y=4$.

One.III.2.27 Define linear systems to be equivalent if their augmented matrices are row equivalent. The proof that equivalent systems have the same solution set is easy.

One.III.2.28 **(a)** The three possible row swaps are easy, as are the three possible rescalings. One of the six possible pivots is $k\rho_1+\rho_2$:

$$\begin{pmatrix}1&2&3\\k\cdot1+3&k\cdot2+0&k\cdot3+3\\1&4&5\end{pmatrix}$$

and again the first and second columns add to the third. The other five pivots are similar.

(b) The obvious conjecture is that row operations do not change linear relationships among columns.

(c) A case-by-case proof follows the sketch given in the first item.

Topic: Computer Algebra Systems

1 **(a)** The commands

```
> A:=array( [[40,15],
            [-50,25]] );
> u:=array([100,50]);
> linsolve(A,u);
```

yield the answer $[1,4]$.

(b) Here there is a free variable:

```
> A:=array( [[7,0,-7,0],
             [8,1,-5,2],
             [0,1,-3,0],
             [0,3,-6,-1]] );
> u:=array([0,0,0,0]);
> linsolve(A,u);
```

prompts the reply $[\_t_1,3\_t_1,\_t_1,3\_t_1]$.

2 These are easy to type in. For instance, the first

```
> A:=array( [[2,2],
             [1,-4]] );
> u:=array([5,0]);
> linsolve(A,u);
```

gives the expected answer of $[2,1/2]$. The others are entered similarly.