LARSON—MATH 310—CLASSROOM WORKSHEET 39
Singular Value Decomposition.

Given any $m \times n$ matrix $A$ we can find orthogonal matrices $U$ and $V$ and a diagonal matrix $\Sigma$ so that $A = U\Sigma V^T$. This is it, the SVD.

Furthermore $V = [\vec{v}_1 \ldots \vec{v}_r \ldots \vec{v}_n]$ (where $r$ is the rank of $A$), $\Sigma$ is an $m \times n$ 0s matrix with positive numbers $\sigma_1, \ldots, \sigma_r$ on the diagonal, $A\vec{v}_i = \sigma_i \vec{v}_i$, $\sigma_i = ||A\vec{v}_i||$, and $U = [\vec{u}_1 \ldots \vec{u}_r \ldots \vec{u}_n]$, where $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ for $i = 1, \ldots r$.

The $\sigma_i$'s are singular values.

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

1. $A$ is an $m \times n$ matrix. Find $m$ and $n$. The following algorithm works for any $m \times n$ matrix!

2. Find the rank $r$ of $A$. (This tells you how many vectors are in the row space and null space of $A$). We proved that this is also the rank of $A^T A$ and $AA^T$.

3. Find $A^T A$.

4. Find the eigenvalues of $A^T A$. There will be $r$ positive eigenvalues: $\sigma^2_1, \ldots, \sigma^2_r$.

5. Find the corresponding eigenvectors for these eigenvalues (of $A^T A$) and normalize them. Call these: $\vec{v}_1, \ldots, \vec{v}_r$. (We proved in class that they must be orthogonal).

6. Normalize the vectors corresponding to the 0-eigenvalues of $A^T A$. Call these $\vec{v}_{r+1}, \ldots, \vec{v}_n$. (In the general case you need to use Gram-Schmidt to find an orthonormal basis of these.)
7. Let \( V = \begin{bmatrix} v_1 \ldots v_r \ldots v_n \end{bmatrix} \).

8. Show that \( V \) is orthogonal.

9. For each \( i \in \{1, \ldots, r\} \), find \( \|A\vec{v}_i\| \). Check that \( \sigma_i = \|A\vec{v}_i\| \).

10. Let \( \Sigma \) be the \( m \times n \) matrix with the \( \sigma_i \)'s on the diagonal for \( i = 1, \ldots, r \), and 0 for every other entry.

11. Find \( AA^T \).

12. Find the eigenvalues of \( AA^T \). There will be \( r \) positive eigenvalues and they should be the same as the ones for \( A^T A \): \( \sigma_1^2, \ldots, \sigma_r^2 \).

13. Find the corresponding eigenvectors for these eigenvalues (of \( AA^T \)) and normalize them. Call these: \( \vec{u}_1, \ldots, \vec{u}_r \). (We proved in class that they must be orthogonal).

14. Let \( U = \begin{bmatrix} u_1 \ldots u_r \ldots u_m \end{bmatrix} \).

15. Show that \( U \) is orthogonal.

16. For each \( i = 1, \ldots, r \), check that \( A\vec{v}_i = \sigma_i \vec{u}_i \).

17. Show that \( A = U\Sigma V^T \).