We found that, for any matrix $A$ (with linearly independent columns), that the projection $\vec{p}$ of a vector $\vec{b}$ onto the column space of $A$ is:

$$\vec{p} = A\vec{x}$$

where

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}.$$ 

so

$$\vec{p} = A(A^T A)^{-1} A^T \vec{b}.$$ 

We can use this idea repeatedly to convert a collection of linearly independent vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ (which are a basis for the space those vectors span) to a nice basis for the same space: orthonormal vectors $\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_k'$.

Here’s the idea informally:

1. Unit-ize $\vec{a}_1$ to get $\vec{a}_1'$.
2. Project vector $\vec{a}_2$ onto the matrix $A$ with $\vec{a}_1'$ as its only column.
3. Let $\vec{a}_2'$ be the error vector $\vec{e} = \vec{a}_2 - \text{projection on } A$ of $\vec{a}_2$, and unit-ize.
4. Repeat as necessary. Project vector $\vec{a}_j$ onto the matrix $A$ with $\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_{j-1}'$ as its columns.
5. Let $\vec{a}_j'$ be the error vector $\vec{e} = \vec{a}_j - \text{projection on } A$ of $\vec{a}_j$, and unit-ize.

Here’s a formal algorithm:

1. Let $\vec{a}_1' = \frac{\vec{a}_1}{||\vec{a}_1||}$. Let $A$ have $\vec{a}_1'$ as its only column.
2. Let $\vec{p} = A(A^T A)^{-1} A^T \vec{a}_2$. Let $\vec{e} = \vec{a}_2 - \vec{p}$. Let $\vec{a}_2' = \frac{\vec{e}}{||\vec{e}||}$.
3. Repeat as necessary. Let $A$ be the matrix with columns $\vec{a}_1', \ldots, \vec{a}_{j-1}'$.
4. Let $\vec{p} = A(A^T A)^{-1} A^T \vec{a}_j$. Let $\vec{e} = \vec{a}_j - \vec{p}$. Let $\vec{a}_j' = \frac{\vec{e}}{||\vec{e}||}$.

Now we’ll find an orthogonal basis for the vector space spanned by vectors $\vec{a}_1 = (1, 0, 0)$ and $\vec{a}_2 = (1, 0, 1)$ and $\vec{a}_3 = (0, 1, 1)$. 
We will find an orthogonal basis for the vector space spanned by vectors \( \vec{a}_1 = (1, 0, 0) \) and \( \vec{a}_2 = (1, 0, 1) \) and \( \vec{a}_3 = (0, 1, 1) \).

1. Find \( \vec{a}_1' \) and find \( A \).

2. Find the projection \( \vec{p} \) of \( \vec{a}_2' \) on the column space of \( A \), then \( \vec{e} \), then \( \vec{a}_2' \).

3. Let \( A \) be the matrix with columns \( \vec{a}_1', \vec{a}_2' \).

4. Find the projection \( \vec{p} \) of \( \vec{a}_3' \) on the column space of \( A \), then \( \vec{e} \), then \( \vec{a}_3' \).

5. Let \( Q \) be the matrix whose columns are \( \vec{a}_1', \vec{a}_2', \vec{a}_3' \). This is an orthogonal matrix. Check that \( Q^TQ = I \).