Orthogonal Basis.

We will find an orthogonal basis for the vector space spanned by vectors \( \vec{a}_1 = (3, 4, 4) \) and \( \vec{a}_2 = (2, 2, 1) \).

Let \( \vec{p} \) be the projection of \( \vec{a}_2 \) on vector \( \vec{a}_1 \). Then \( \vec{a}_2' = \vec{a}_2 - \vec{p} \) is orthogonal to \( \vec{a}_1 \).

We found that, for any matrix \( A \) (with linearly independent columns), that the projection \( \vec{p} \) of a vector \( \vec{b} \) onto the column space of \( A \) is:

\[
\vec{p} = A \vec{x} \text{ where } \vec{x} = (A^T A)^{-1} A^T \vec{b}.
\]

so \( \vec{p} = A(A^T A)^{-1} A^T \vec{b} \).

1. Let \( A \) be the matrix formed by the single column \( \vec{a}_1 \).

2. Find \( A^T \).

3. Find \( A^T A \).

4. Find \( (A^T A)^{-1} \).
5. Find \((A^T A)^{-1} A^T\).

6. Find \((A^T A)^{-1} A^T \vec{a}_2\).

7. Find \(A(A^T A)^{-1} A^T \vec{a}_2\).

8. Find \(\vec{a}_2' = \vec{a}_2 - A(A^T A)^{-1} A^T \vec{a}_2\).

9. Check that \(\vec{a}_2'\) and \(\vec{a}_1\) are orthogonal. So \(\vec{a}_1, \vec{a}_2'\) are an orthogonal basis for the space spanned by the original vectors \(\vec{a}_1, \vec{a}_2\).

Now we will find an orthnormal basis by normalizing these vectors.

10. Find \(|\vec{a}_1|\) and \(|\vec{a}_1|\), and \(|\vec{a}_2'|\) and \(|\vec{a}_2'|\).

11. Write a matrix \(Q\) whose columns are \(\frac{\vec{a}_1}{||\vec{a}_1||}\) and \(\frac{\vec{a}_2'}{||\vec{a}_2'||}\). This is an orthogonal matrix. Check that \(Q^T Q = I\).