Math 305 - Homework #1

(1) <u>Show</u>: $\gcd(a,b) = \gcd(a, a-b)$

Let $d_1 = \gcd(a,b)$

$d_2 = \gcd(a, a-b)$

<u>Note</u>: It is enough to show that $d_1 \leq d_2$ and $d_2 \leq d_1$.

$d_1 = \gcd(a,b) \Rightarrow d_1 \mid a$ and $d_1 \mid b$

Thus, $\exists\, a', b' \in \mathbb{Z}$ such that

$d_1 a' = a$ and

$d_1 b' = b$

so, $d_1 a' - d_1 b' = a - b$,

and $d_1 (a' - b') = a - b$,

so $d_1 \mid (a-b)$

Thus $d_1 \leq d_2$

$d_2 = \gcd(a, a-b) \Rightarrow d_2 \mid a$ and
$d_2 \mid (a-b)$

so $\exists\, a', c \in \mathbb{Z}$ such that

$d_2 a' = a$ and

$d_2 c = a - b$

Then $d_2 c - d_2 a' = (a-b) - a$

$d_2 (c - a') = -b$

or $d_2 (a' - c) = b$

Thus $d_2 \mid b$

Since $d_2 \mid a$ and $d_2 \mid b$

It follows that $d_2 \leq d_1$ ✓

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(2) & (3)

we know: $\gcd(a,b) = \gcd(b,a)$

and by our previous work that

$\gcd(b,a) = \gcd(b, a-b)$

We will use the general form:

$\gcd(x,y) = \gcd(x, x-y)$

Now repeatedly.

If we let

$$x = b$$
$$y = a - b$$

we get

$$\gcd(b, a-b) = \gcd(b, (a-b)-b)$$
$$= \gcd(b, a-2b)$$

If we now let

$$x = b$$
$$y = a - 2b$$

we get

$$\gcd(b, a-2b) = \gcd(b, a-3b)$$

so, $\gcd(b,a) = \gcd(b, a-2b) = \gcd(b, a-3b)$

and we can continue

until we have

$$\gcd(b,a) = \gcd(b, a-2b) = \ldots$$
$$= \gcd(b, a-kb) \text{ for any } k$$

but we wanted

$$\gcd(a,b) = \gcd(a, a-kb)$$

$$\gcd(a,b) \overset{?}{=} \gcd(b, a-kb)$$

$$\gcd(b,a) \quad \checkmark$$

So, it remains to show that

$$\gcd(a, a-kb) = \gcd(b, a-kb)$$

First check that

$$\gcd(a, a-2b) = \gcd(b, a-2b)$$

This can then be used to show

$$\gcd(a, a-3b) = \gcd(b, a-3b)$$

In general,

$$\gcd(a, a-cb) = \gcd(b, a-cb)$$

can be used to show that

$$\gcd(a, a-(c+1)b) = \gcd(b, a-(c+1)b)$$

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(4) Done in class

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(5) $\gcd(\overset{a}{100}, \overset{b}{47})$

$= \gcd(47, 6)$ as $100 = 47 \cdot 2 + 6$ r

$= \gcd(6, 5)$ as $47 = 6 \cdot 7 + 5$ r

$= \gcd(5, 1)$ as $6 = 5 \cdot 1 + 1$ r

$= 1$