1) Show: \( \gcd(a, b) = \gcd(a, a-b) \)

Let \( d_1 = \gcd(a, b) \)
\[ d_2 = \gcd(a, a-b) \]

Note: It is enough to show that \( d_1 \leq d_2 \) and \( d_2 \leq d_1 \).

\[ d_1 = \gcd(a, b) \Rightarrow d_1 \mid a \quad \text{and} \quad d_1 \mid b \]

Thus, \( \exists a', b' \in \mathbb{Z} \) such that \( d_1 a' = a \) and \( d_1 b' = b \).

So, \( d_1(a' - b') = a - b \), and \( d_1(a' - b') = a - b \),

So, \( d_1 \mid (a - b) \)

Thus, \( d_1 \leq d_2 \).
\[ d_2 = \gcd(a, a-b) \Rightarrow d_2 \mid a \text{ and } d_2 \mid (a-b) \]

So \( \exists a', c \in \mathbb{Z} \text{ such that } \]
\[ d_2 a' = a \text{ and } \]
\[ d_2 c = a - b \]

Then \( d_2 (c - a') = (a - b) - a \)
\[ d_2 (c - a') = -b \]

or \( d_2 (a' - c) = b \)

Thus \( d_2 \mid b \)

Since \( d_2 \mid a \) and \( d_2 \mid b \)
It follows that \( d_2 \leq d_1 \)

(2) & (3)

We know: \( \gcd(a, b) = \gcd(b, a) \)

and by our previous work that
\[ \gcd(b, a) = \gcd(b, a-b) \]

We will use the general form:
\[ \gcd(x, y) = \gcd(x, x-y) \]
Now repeatedly.

If we let
\[ x = b \]
\[ y = a - b \]
we get
\[ \gcd(b, a-b) = \gcd(b, (a-b) - b) = \gcd(b, a - 2b) \]

If we now let
\[ x = b \]
\[ y = a - 2b \]
we get
\[ \gcd(b, a-2b) = \gcd(b, a - 3b) \]

So, \( \gcd(b, a) = \gcd(b, a-2b) = \gcd(b, a-3b) \)
and we can continue

Until we have
\[ \gcd(b, a) = \gcd(b, a-2b) = \ldots = \gcd(b, a - kb) \text{ for any } k \]

but we wanted
\[ \gcd(a, b) = \gcd(a, a - kb) \]
\[ \checkmark \quad \checkmark \quad \checkmark \quad \checkmark \quad \checkmark \]

\[ \gcd(b, a) \quad \gcd(b, a - kb) \]
So, it remains to show that
\[ \gcd(a, a - kb) = \gcd(b, a - kb) \]

First check that
\[ \gcd(a, a - 2b) = \gcd(b, a - 2b) \]

This can then be used to show
\[ \gcd(a, a - 3b) = \gcd(b, a - 3b) \]

In general,
\[ \gcd(a, a - cb) = \gcd(b, a - cb) \]

Can be used to show that
\[ \gcd(0, a - (c+1)b) = \gcd(b, a - (c+1)b) \]

\[ \text{(4) Done in class} \]

\[ \begin{align*}
\text{gcd}(100, 47) &= \text{gcd}(47, 6) \\
&= \text{gcd}(6, 5) \\
&= \text{gcd}(5, 1) \\
&= 1
\end{align*} \]