Last name _____

First name _____

LARSON—MATH 305—CLASSROOM WORKSHEET 40
Sage LAB—Quadratic Reciprocity!

Organizational Notes

1. A Zoom recording link and class notes will be sent out after each Zoom class.
2. Mon., Nov. 23 is the last class day.
3. Test #2 is Mon., Nov. 30.
4. Don't forget your Teacher Evaluations. They are important!

Logging into Sage/CoCalc

1. Start the Chrome browser.
2. Go to `http://cocalc.com` and sign in.
3. You should see an existing Project for our class. Click on that.
4. Click "New", call it **c40**, then click "Sage Worksheet".
5. For each problem number, label it in the Sage cell where the work is. So for Problem 1, the first line of the cell should be `#Problem 1`.
6. When you are finished with the worksheet, click "make pdf", email me the pdf (at `clarson@vcu.edu`, with a header that says **Math 305 c40 worksheet attached**).

Our ultimate goal is to prove:

(Gauss' Quadratic Reciprocity Law) For distinct odd primes p and q,

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\left(\frac{p}{q}\right)$$

Review

1. **(Gauss' Lemma)**. Let p be a prime, and $a \in \mathbb{Z}$ coprime to p. Consider $S_1 = \{a, 2a, 3a, \ldots, \frac{p-1}{2}a\}$. Let S_2 be the set of residues in S_1 in $(0,p) \cap \mathbb{Z}$. Let S_3 be the residues of S_1 in $(-\frac{p}{2}, \frac{p}{2}) \cap \mathbb{Z}$ (which must be the same as the residues of S_2 in $(-\frac{p}{2}, \frac{p}{2}) \cap \mathbb{Z}$). Let ν be the number of negative elements in S_3. Then, $\left(\frac{a}{p}\right) = (-1)^{\nu}$.

2. (Lemma 4.3.3) Let $a, b \in \mathbb{Q}$. Then for any integer n,

$$\#(a,b) \cap \mathbb{Z} \equiv \#(a, b+2n) \cap \mathbb{Z} \pmod 2$$

and

$$\#(a,b) \cap \mathbb{Z} \equiv \#(a-2n, b) \cap \mathbb{Z} \pmod 2$$

provided that each interval in the congruence is non-empty.

Experiments

We'll do experiments today to investigate $\left(\frac{2}{p}\right)$ and to check that, for odd primes p and q that differ by a multiple of 4, and any integer a, $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$.

1. **(Legendre Symbol of 2)**. What is $\left(\frac{2}{p}\right)$? Find this for prime p for $p = 3..47$. Is there a pattern?

2. Check that: check that, for odd primes p and q that differ by a multiple of 4, and any integer a,

$$\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right).$$

3. Try $p = 3$ and $q = 7$, $p = 5$ and $q = 13$.

4. **(Legendre Symbol of 2)**. Show:

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8 \\ -1 & \text{if } p \equiv \pm 3 \pmod 8 \end{cases}$$

Proof Ideas:

(a) Gauss' Lemma says $\left(\frac{2}{p}\right) = (-1)^{\nu}$. So we need to find ν.

(b) Let $S = \{1 \cdot 2, 2 \cdot 2, \ldots, \frac{p-1}{2} \cdot 2\}$.

(c) Note that $\#(S \cap (-\frac{p}{2}, \frac{p}{2})) = \#(S \cap (\frac{p}{2}, p)) = \{1, \ldots, \frac{p-1}{2}\} \cap (\frac{p}{4}, \frac{p}{2})) = \#(\mathbb{Z} \cap (\frac{p}{4}, \frac{p}{2}))$.

(d) $p = 8c + r$, where $r = 1, 3, 5, 7$. How many integers are in $(\frac{p}{4}, \frac{p}{2})$ in each case?

(e) Find ν in each case and apply Gauss' Lemma.

To see what is going on, try $p = 7$. Find S, $\#(S \cap (\frac{p}{2}, p))$, $\#(\mathbb{Z} \cap (\frac{p}{4}, \frac{p}{2}))$, ν and $\left(\frac{2}{p}\right)$.