Last name ______________________

First name ______________________

LARSON—MATH 305—CLASSROOM WORKSHEET 14
Sage LAB!—Rings &Units

Organizational Notes

1. A Zoom recording link and class notes will be sent out after each Zoom class.
2. Remember to send your answers to the classroom worksheets, and pdfs of your Lab work. Title your email with enough to help me record your "participation".

Logging into Sage/CoCalc

1. Start the Chrome browser.
2. Go to `http://cocalc.com` and sign in.
3. You should see an existing Project for our class. Click on that.
4. Click "New", call it **c14**, then click "Sage Worksheet".
5. For each problem number, label it in the Sage cell where the work is. So for Problem 1, the first line of the cell should be `#Problem 1`.
6. When you are finished with the worksheet, click "make pdf", email me the pdf (at `clarson@vcu.edu`, with a header that says **Math 305 c14 worksheet attached**).

From our Book

1. Define the ring $\mathbb{Z}/3\mathbb{Z}$. Call it $R3$. Run: `R3 = Integers(3)`.
2. List the elements of $R3$ to see what we have. Run: `list(R3)`.

   A ring R is a *field* if there is a *multiplicative identity* (or *one*) and every element has a *multiplicative inverse*, that is, for every non-zero $a \in R$ there is a $b \in R$ such that $a * b = 1$.

   We'll see that for a prime p, $\mathbb{Z}/p\mathbb{Z}$ is not only a ring but that it is (importantly) a field.

3. Do some experiments to see how the elements of $R3$ multiply. Check that every non-zero element has a multiplicative inverse.
4. Now let's define $\mathbb{Z}/5\mathbb{Z}$. Do some experiments to show that this ring is indeed a field.

5. Now let's define $\mathbb{Z}/6\mathbb{Z}$. Do some experiments to show that this ring is *not* a field.

6. Here we see an important fact: 2 does not have a multiplicative inverse in $R6$. Why?

   Here's what we see: for a in $\mathbb{Z}/n\mathbb{Z}$ if there is a b in $\mathbb{Z}/n\mathbb{Z}$ with $a * b = 1$ then a does not have a multiplicative inverse.

7. This isn't the only instance. Let's look at 9 in $\mathbb{Z}/12\mathbb{Z}$. Check that it doesn't have a multiplicative inverse. Why?

8. Let's look at one more example of a ring with prime modulus: $\mathbb{Z}/7\mathbb{Z}$. For each element in $\mathbb{Z}/7\mathbb{Z}$, let's list its products with the non-zero elements of $\mathbb{Z}/7\mathbb{Z}$. Do you notice anything interesting?

9. Now we can finish our explanation: in $\mathbb{Z}/p\mathbb{Z}$ (where p is prime), and a is any element, the products of a with the non-zero elements are all different. Why?

10. What is the problem in $\mathbb{Z}/12\mathbb{Z}$? 9 doesn't have a multiplicative inverse—but isn't a divisor of 12. We can still show that there is a b in $\mathbb{Z}/12\mathbb{Z}$ so that $9 * b = 0$ (which was the problem in $\mathbb{Z}/6\mathbb{Z}$). List all the products of $9 * b$ with non-zero b.

    We actually have a *proof* now that $\mathbb{Z}/p\mathbb{Z}$ is a field if and only if p is prime.

    The elements of $\mathbb{Z}/n\mathbb{Z}$ that have multiplicative inverses are called *units* in $\mathbb{Z}/n\mathbb{Z}$. So every non-zero element in $\mathbb{Z}/p\mathbb{Z}$ is a unit.

11. Find all the units in $\mathbb{Z}/6\mathbb{Z}$.

12. Find all the units in $\mathbb{Z}/12\mathbb{Z}$.

13. Can you conjecture a condition for when a in $\mathbb{Z}/n\mathbb{Z}$ is a unit?

    Extras

14. If p and $p+2$ are both prime, they are called *twin primes*. Find the first several pairs of twin primes.