1. Log in to your Sage/Cocalc account.
   
   (a) Start the Chrome browser.
   (b) Go to http://cocalc.com and sign in.
   (c) You should see an existing Project for our class. Click on that.
   (d) Click “New”, call it c16, then click “Sage Worksheet”.

   It is often intuitive to define a function recursively, but usually the same function can be defined without recursion.

2. Define a non-recursive (iterative) function \( \text{fibonacci2}(n) \) that computes the \( n^{th} \) Fibonacci number.

3. Evaluate and write down what you get for \( \text{timeit("fibonacci2(10)"), timeit("fibonacci2(20)"), and timeit("fibonacci2(25)").} \)

4. The recursive \( \text{fibonacci}(n) \) function we defined takes a very long time to respond for \( n = 30 \) and may never respond for \( n = 40 \). Now try \( \text{fibonacci2}(40) \) and \( \text{fibonacci2}(400) \). Why does the iterative function work while the recursive function does not?

5. Solve the equation \( \frac{a+b}{a} = \frac{b}{a} \), for \( a \) and \( b \). Find \( \frac{a}{b} \). Get a 10-digit approximation for this quantity (this is the Golden Ratio).
6. Define a function \( \text{fib\_ratio}(n) \) which returns the ratio of the \((n+1)\)th Fibonacci number to the \(n\)th. Find \( \text{fib\_ratio}(10) \) and \( \text{fib\_ratio}(100) \). Compare this answer to your previous answer. What can you conjecture?

Riemann Integration

Given a continuous function \( f(x) \) on an interval \([a, b]\) we want to find the \textit{area} between the curve, the \(x\)-axis and the lines \( y = a \) and \( y = b \). One way to do this is to use the Fundamental Theorem of Calculus and integrate. Unfortunately, it is difficult to find anti-derivatives for many (most) functions. So we need a different approach to get at least an approximate integral.

One way to do this is to slice up \([a, b]\) into \(n\) equal-sized intervals \([a_0, a_1], [a_1, a_2], \ldots, [a_n, a_{n+1}]\) (where \(a_1 = a\) and \(a_{n+1} = b\)), pick a point \(c_i\) from each interval \([a_i, a_{i+1}]\) and compute the area \( f(c_i) \cdot \Delta \) of a rectangle, where \(\Delta\) is the interval length \(a_{i+1} - a_i\).

There are different ways to pick the \(c_i\)'s. You could pick the leftmost point of the interval, the midpoint, the rightmost point, or even a random point.

The \textit{Riemann Integral} is defined to be the \textit{limit} of these area approximations as \(n\) goes to infinity of this quantity.

7. Here is a function \texttt{leftpoint\_riemann}(f,a,b,n) which computes the leftpoint Riemann sums for \(n\) equal intervals.

```python
def leftpoint_riemann(f,a,b,n):
    area=0
    Delta=(b-a)/n
    for i in [0..(n-1)]:
        leftpoint=a+i*Delta
        area=area+f(leftpoint)*Delta
    return area
```

Find the value of \texttt{leftpoint\_riemann}(f,a,b,n) for \(f(x)=x^2\) on \([0,3]\) with \(n = 2\), \(n = 5\), \(n = 10\) and \(n = 100\). Here you are making the intervals smaller and smaller, giving a better and better approximation.

8. Given a continuous function \(f(x)\) on \([a,b]\), define a function \texttt{rightpoint\_riemann}(f,a,b,n) which computes the rightpoint Riemann sums for \(n\) equal intervals. Find the values of \texttt{rightpoint\_riemann}(f,a,b,n) for \(f(x)=x^2\) on \([0,3]\) with \(n = 2\), \(n = 5\), \(n = 10\) and \(n = 100\).