ULTRAPRODUCTS, THE COMPACTNESS THEOREM AND APPLICATIONS

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1. About

This is a set of lecture notes used in a course on model theory at Virginia Commonwealth University (Math 591 - Topics: Logic and Mathematical Structures), which I taught jointly with Sean Cox in the spring of 2014.

Most of the material contained in these notes can be found in the following sources.

References:

- Chang and Keisler. Model Theory, *Studies in Logic and the Foundations of Mathematics, Volume 73*, Third Ed., 1990, North Holland.
- Endertion. A Mathematical Introduction to Logic, Second Ed., 2001, Harcourt Academic Press.
- Rothmaler. Introduction to Model Theory *Algebra, Logic and Applications Series, Volume 15*, First Ed., 2000, Gordon and Breach.

2. Introduction

In courses on first-order logic, one usually derives the compactness theorem from Gödel's completeness theorem, which says that every logical consequence can be derived in a certain effective formal proof system. The

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compactness theorem then follows immediately from the finite character of that proof system ("every formal proof is finite"). We will show that, in fact, the finite character of logical consequence (i.e. the compactness theorem) can be derived without any reference to formal systems or formal proof. Along the way we will discuss the Axiom of Choice, filters, ultrafilters, ultraproducts, non-standard models of arithmetic, and Łoś theorem.

We will then give some applications of the compactness theorem. In particular, we will show that the upward Löwenheim-Skolem theorem follows quite easily from the compactness theorem. We will also discuss how all of this leads to nonstandard models of arithmetic (i.e. structures $\mathcal{M}$ satisfying all the same sentences as the standard model $(\mathbb{N}, 0, 1, +, \cdot, \leq)$ with $\mathbb{N} \subsetneq M$ which contain elements $a \in M$ such that $n \leq a$ for all $n \in \mathbb{N}$). The notes conclude with a discussion of types, saturation of ultraproducts and a simplified version of a famous theorem of Keisler and Shelah.

3. Background material on orderings

An **ordering** is a first-order structure of the form $(P, \leq)$ where P is a set and $\leq$ is a binary relation that satisfies certain properties depending on the type of ordering. A typical example of an ordering is the set of natural numbers $\mathbb{N}$ with the usual ordering $\leq^{\mathbb{N}}$; for example, $0 \leq 4$, $17 \leq 28$, and $15 \not\leq 8$.

Let's try to specify some of the important properties of $(\mathbb{N}, \leq)$.

- (Totality) For any numbers $n, m \in \mathbb{N}$, either $n \leq m$ or $m \leq n$.
- (Antisymmetry) For any numbers $n, m \in \mathbb{N}$, if $n \leq m$ and $m \leq n$ then $n = m$.
- (Transitivity) For any numbers $n, m, k \in \mathbb{N}$, if $n \leq m$ and $m \leq k$ then $n \leq k$.

These three properties define what we call a "linear order."

Definition 1. A **linear order** is a structure of the form $(I, \leq)$ satisfying the following three properties, for all $a, b, c \in I$.

(1) (Totality) $a \leq b$ or $b \leq a$.
(2) (Antisymmetry) $a \leq b$ and $b \leq a$ implies $a = b$.
(3) (Transitivity) $a \leq b$ and $b \leq c$ implies $a \leq c$.

Example 1. The following structures are linear orders: $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, and $(\mathbb{R}, \leq)$.

Definition 2. A **partial order** is a structure of the form $(P, \leq)$ satisfying the following three properties, for all $a, b, c \in P$.

(1) (Reflexivity) $a \leq a$.
(2) (Antisymmetry) $a \leq b$ and $b \leq a$ implies $a = b$.
(3) (Transitivity) $a \leq b$ and $b \leq c$ implies $a \leq c$.

Example 2. Let's show that the collection of all subsets of $\mathbb{N}$, written as $P(\mathbb{N})$, together with the subset ordering '$\subseteq$' is a partial order but not a linear order. In other words, let us show that $(P(\mathbb{N}), \subseteq)$ is a partial order but not a linear order. (Reflexive) If $A \in P(\mathbb{N})$ then $A \subseteq \mathbb{N}$ and clearly $A \subseteq A$. (Antisymmetric) Suppose $A, B \in P(\mathbb{N})$. If $A \subseteq B$ and $B \subseteq A$ then of course $A = B$. (Transitivity) Suppose $A, B, C \in P(\mathbb{N})$. Then if $A \subseteq B$ and $B \subseteq C$ then clearly $A \subseteq C$.

Why is $(P(\mathbb{N}), \subseteq)$ not a linear order? It must be because the totality condition is not true in $(P(\mathbb{N}), \subseteq)$. We need to find two subsets $A, B \subseteq N$ such that both $A \subseteq B$ and $B \subseteq A$ are false. Just take $A = \{5\}$ and $B = \{17\}$. Then $A \not\subseteq B$ and $B \not\subseteq A$, so the Totality condition fails. Indeed, any subseteq $A, B \subseteq \mathbb{N}$ that are disjoint would work. This shows that $(\mathbb{P}(\mathbb{N}), \subseteq)$ is not a linear order.

Definition 3. If $(P, \leq)$ is a partial order and $A \subseteq P$ is nonempty, then $a \in P$ is called the **least element of** A if $a \in A$ and for all $b \in A$ we have $a \leq b$.

An element $a \in P$ is called the **greatest element of** A if $a \in A$ and for all $b \in A$ we have $b \leq a$.

Definition 4. We say that a linear order $(I, \leq)$ is a **well-order** if every nonempty subset $A \subseteq I$ has a least element. In other words, $(I, \leq)$ is a well-order if it is a linear order and for every subset $X \subseteq I$ there is an element $a \in X$ such that for all $b \in X$ we have $a \leq b$.

Example 3. Not every linear order is a well-order. Consider the integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

with their standard order $(\mathbb{Z}, \leq)$. Clearly $\mathbb{Z}$ has no least element, and so $(\mathbb{Z}, \leq)$ is not a well-order. Some other examples of linear orders that are not well-orders: $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$.

Example 4. $(\mathbb{Q}, \leq)$ is not a well-order because many of its subsets have no least element. For example the set $X = \{\ldots, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$ is a subset of $\mathbb{Q}$ with no least element. However, notice that the set X does have a "lower bound" in $(\mathbb{Q}, \leq)$, namely 0: for every natural number $n > 0$ we have $0 \leq \frac{1}{n}$.

Similarly, the set $X = \{0, \frac{1}{2}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$ has an "upper bound," namely 1, but X has no greatest element.

Definition 5. Suppose $(P, \leq)$ is a partial order. We say that $a \in P$ is a **lower bound** of a set $X \subseteq P$ if for every $b \in X$ we have $a \leq b$.

We say that $a \in P$ is an **upper bound** of $X \subseteq P$ if for every $b \in X$ we have $b \leq a$.

Definition 6. Let $(\mathbb{P}, \leq)$ be a partial order and suppose $A \subseteq P$ is nonempty. We say that $a \in A$ is a **minimal element** of A if no elements of A are "smaller" than a. In other words, a is a minimal element of A if for all $b \in A$ with $b \neq a$ we have $b \not\leq a$.

We say that $a \in A$ is a **maximal element** of A if no elements of A are "larger" than a. In other words, a is a maximal element of A if for all $b \in A$ with $b \neq a$ we have $a \not\leq b$.

Example 5. The set $P(\mathbb{N})$ has a $\subseteq$-least element.

Example 6. Consider the set

$$X_0 = \{\{3\}, \{3,5\}, \{3,5,12\}, \{3,5,12,17\}, \{3,5,12,17,22\}\}.$$

The $\subseteq$-least element of X_0 is $\{3\}$ because $\{3\}$ is a subset of all the other sets in X_0. $\{3\}$ is also a minimal element of X_0 and a lower bound of X_0.

Example 7. Let us now consider the collection

$$X_1 = \{\{3\}, \{7\}, \{3,5\}, \{7,8\}, \{3,5,7,8\}\}.$$

Now X_1 does not have a $\subseteq$-least element because none of the sets in X_0 are subsets of all the others. However, X_1 does have a $\underline{\subseteq\text{-minimal}}$ element—in fact, it has two $\subseteq$-minimal elements. $\{3\}$ is a $\subseteq$-minimal element because for every other $a \in X_1$ with $a \neq \{3\}$, we have $a \not\subseteq \{3\}$. Similarly, $\{7\}$ is a $\subseteq$-minimal element of X_1.

Is there a subset of $\mathbb{N}$ that is a lower bound of X_1? Answer: yes. The emptyset $\emptyset$ is a subset of $\mathbb{N}$ and $\emptyset$ is a subset of every set. So in particular, for every $a \in X_1$ we have $\emptyset \subseteq a$.

Does X_1 have a greatest element? Answer: yes. $\{3,5,7,8\}$ is the greatest element of X_1 because it contains all the others.

Example 8. Consider the set

$$X_2 = \{\{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{7\}, \{8\}, \{7,8\}\}.$$

Notice that X_2 does not have a $\subseteq$-greatest element since none of the sets in X_2 contain all the others. However, X_2 does have a $\subseteq$-maximal element—in fact, it has two maximal elements. Does X_2 have an upper bound in $(P(\mathbb{N}), \subseteq)$?

Definition 7. Suppose $(P, \leq)$ is a partially ordered set. We call $T \subseteq P$ a **chain** if T is linearly ordered by $\leq$.

Example 9. Consider the structure $(\mathbb{R}, \leq)$. Then $\mathbb{Z} \subseteq \mathbb{R}$ is a chain. Does the chain have an upper bound in $(\mathbb{R}, \leq)$?

Example 10. Consider the partial order $(P(\mathbb{N}), \subseteq)$. The following is a chain:

$$\{2\} \subseteq \{2,4\} \subseteq \{2,4,6\} \subseteq \cdots$$

This chain has a least element, but does not have a greatest element. The chain does have an upper bound in $(P(\mathbb{N}), \subseteq)$.

Example 11. Is there a chain in $(P(\mathbb{N}), \subseteq)$ that has no least element? Answer: Yes. For each $n \in \mathbb{N}$ let $I_n = [n, \infty) = \{m \in \mathbb{N} \mid n \leq m\}$ be the standard interval from n to ∞. Then the collection $\{I_n \mid n \in \mathbb{N}\}$ has no $\subseteq$-least element:

$$\cdots I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq I_0 = \mathbb{N}. \tag{3.1}$$

The collection does have a lower bound in $(P(\mathbb{N}), \subseteq)$: the empty set is a subset of every set.

The Axiom of Choice.

If A is a nonempty set, then A has at least one element, then we can always choose $x \in A$. Indeed, if $A_0, A_1, \ldots, A_n$ are finitely many nonempty sets, then we can always choose $x_0 \in A_0$, $x_1 \in A_1$, …, $x_n \in A_n$. In other words, there is a function f such that $f(A_i) = x_i \in A_i$ for each i with $0 \leq i \leq n$. But, what if we have an infinite set $\{A_0, A_1, A_2, \ldots\}$ such that for each $n \in \mathbb{N}$ the set A_n is nonempty. Is there a function f such that for each $i \in \mathbb{N}$ we have $f(A_i) \in A_i$? It turns out that if we want to be able to prove that such a function exists, we must use the Axiom of Choice.

> **The Axiom of Choice:** If X is a set of nonempty sets then there exists a function f with domain X such that for every $a \in X$, $f(a) \in a$.

The Axiom of Choice is 'independent' from the other foundational axioms of set theory ZF, meaning that set theory ZF cannot prove the Axiom of Choice, nor can it prove the negation of the Axiom of Choice.

The Axiom of choice is equivalent to the well-ordering principle stated below.

> **The Well-ordering Principle:** Every set X can be well-ordered. In other words, for every set X there is a well-order $\leq$ of X.

Well-orders $(X, \leq)$ have the property that given any element $a \in X$, which is not the greatest element of X, there is a least element greater than a, or a "next element after a." The Well-ordering Principle (and hence the Axiom of Choice because they are equivalent) implies that the set of real numbers $\mathbb{R}$ can be well-ordered. However, there is no canonical way of defining a well-order on $\mathbb{R}$. Try to think about how you would do this. What is the least element in the well-order? Let's say the least element is 0. What should the next element be? 1? π? $0.272727272\ldots$?

Another important, and widely used equivalent formulation of the Axiom of Choice is Zorn's Lemma. We will use Zorn's Lemma to show that "ultrafilters" exist.

Proposition 1. *The following are equivalent.*

1. ***The Axiom of Choice***
2. ***Zorn's Lemma:*** *If $(P, \leq)$ is a nonempty partially ordered set and every chain $T \subseteq P$ has an upper bound $u \in P$, then P contains a maximal element.*

Proof. I will only provide a proof *sketch* of (1) $\implies$ (2). Let us assume (1) and $\neg$ (2) and derive a contradiction. Since $\neg$ (2) holds, there is a partially ordered set $(P, \leq)$ such that every chain $T \subseteq P$ has an upper bound and yet P has no maximal element.

First let us show that given any chain $T \subseteq P$, there is an element $p_T \in P$ such that p_T is strictly larger than every element of T. Suppose $T \subseteq P$ is a chain. Then, by our assumption about P, there is an upper bound $u \in P$

of T. Since we are assuming that P has no maximal elements we conclude that there is a $p_T \in P$ such that $u < p_T$.

This shows that for each chain $T \subseteq P$ the set

$$X_T := \{p \in P \mid \forall t \in T \ (t < p)\}$$

is nonempty. Now let $X := \{X_T \mid T \subseteq P \text{ is a chain}\}$ and notice that X is a set of nonempty sets, and so the Axiom of Choice (1) implies that there is a function f such that for every chain $T \subseteq P$ we have $f(X_T) \in X_T$. This means for every chain $T \subseteq P$ we have that $f(X_T) \in P$ is greater than every element of T.

One can now use this function f to define a chain in P:

$$p_0 < p_1 < p_2 < \cdots$$

that gets longer and longer with no end. Eventually the chain will have a "larger size" that P itself, and this is a contradiction.

□

4. Filters and Ultrafilters

Next I will introduce the notion of "filter," which will provide a precise way to say that certain sets are "big" and certain sets are "small." Let us first consider an example. What subsets of the natural numbers $\mathbb{N}$ should be considered big? Well, since $\mathbb{N}$ is infinite, it seems natural to say that finite subsets of $\mathbb{N}$ should not be big, and in fact, we will say that finite sets are small. Should every infinite set be considered big? It will be useful to make a distinction and declare that some infinite subsets of $\mathbb{N}$ are big while others are small. For example, it may be useful to consider the set $\{2n \mid n \in \mathbb{N}\}$ small while at the same time regarding the set $\{n \mid n > 67\}$ to be big. There are some rules that our definition of "bigness" should obey. For example if A is a big set and $A \subseteq B$ then B should be a big set.

Definition 8 (Filter)**.** Let I be a set. A nonempty collection F of subsets of I is called a **filter** (**over** I) if the following conditions hold true.

(1) $\emptyset \notin F$ and $I \in F$.
(2) If $A, B \in F$, then $A \cap B \in F$.
(3) If $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$.

The idea is that if F is a filter and $A \in F$, then A is considered to be big (in the sense of F).

Example 12 (Tail Filter)**.** For each $k \in \mathbb{N}$, let $I_k = \{n \mid n \geq k\} = [k, \infty)$. Let us show that the set $F := \{X \subseteq N \mid \exists k \ (I_k \subseteq X)\}$ is a filter over $\mathbb{N}$. We just need to check the three requirements in the definition.

(1) Of course, $\emptyset \notin F$ because $\emptyset$ does not contain any interval of the form $I_k = [k, \infty)$. Clearly $\mathbb{N} \in F$.
(2) If $A, B \in F$ then there exists $k, m \in \mathbb{N}$ such that $I_k \subseteq A$ and $I_m \subseteq B$. To see that $A \cap B \in F$ we need to check that $A \cap B$ contains an interval of the form I_n. Let $n = \max(k, m)$. Then $I_n = I_k \cap I_m \subseteq A \cap B$.

(3) Suppose $A \in F$ and $A \subseteq B \subseteq \mathbb{N}$. Since $A \in F$ there is a $k \in \mathbb{N}$ such that $I_k \subseteq A \subseteq B$. Hence $B \in F$.

Remark 1. Notice that the tail filter in Example 12 provides a good notion of being "big." Whereas the principle filter $F_n = \{A \subseteq \mathbb{N} : n \in A\}$, where $n \in \mathbb{N}$, does not: the filter F_n tells us that $\{n\}$ is big because $\{n\} \in F_n$, but of course intuitively, the set $\{n\}$ is not big because it has only one element!

Exercise 1 (Frechét filter)**.** Suppose X is an infinite set. Show that the following collection of subsets of X is a filter.

$$F = \{A \subseteq X \mid X \setminus A \text{ is finite}\}$$

Definition 9. Let I be a set. A collection E of subsets of I is said to have the **finite intersection property** if $E \neq \emptyset$ and every intersection of finitely many members of E is nonempty.

Exercise 2. Let I be a set. Show that every filter F over I has the finite intersection property. (Hint: use induction.)

Exercise 3. Show that if E is a collection of subsets of some set I and E has the finite intersection property, then there is a filter $F \supseteq E$ extending E. (Hint: Let F be the collection of all subsets $A \subseteq I$ such that A contain a finite intersection of elements of E.)

Definition 10. We say that U is an **ultrafilter** over a set I if U is a nonempty collection of subsets of I such that the following properties hold.

(1) $\emptyset \notin U$ and $I \in U$.
(2) If $A, B \in U$, then $A \cap B \in U$.
(3) If $A \in U$ and $A \subseteq B \subseteq I$, then $B \in U$.
(4) For every $X \subseteq I$, either $X \in U$ or $I \setminus X \in U$.

Example 13 (Principle Filter)**.** Suppose X is a set and $p \in X$. Then $F_p = \{A \subseteq X : p \in A\}$ is the **principal ultrafilter over X generated by p.** Typically principal ultrafilters are not very interesting.

Lemma 1. *Suppose F is a filter on I. The following are equivalent.*

(i) *F is a **maximal filter** in the sense that whenever $F' \supseteq F$ is a filter one has $F' = F$.*
(ii) *F is an ultrafilter.*

Proof. (ii) $\implies$ (i). This is quite easy. If F is an ultrafilter and $F' \supsetneq F$ is a filter extending F then there is a set $A \in F' \setminus F$. Since $A \notin F$ and F is an ultrafilter, it follows by ultrafilter property (4) that $I \setminus A \in F$. But then $\emptyset = A \cap (I \setminus A) \in F'$, a contradiction.

(i) $\implies$ (ii). Suppose F is a maximal filter. For a contradiction, assume that F is <u>not</u> an ultrafilter. Then there is a subset $X_0 \subseteq I$ such that $X_0 \notin F$ and $I \setminus X_0 \notin F$. One can check that

$$F' := \{X \subseteq I \mid X_0 \subseteq X \text{ or there is a } Y \in F \text{ such that } Y \cap X_0 \subseteq X\}$$

is a filter, which is a contradiction since $F \subsetneq F'$. Let's check that F' is a filter.

(1) First notice that $X_0 \neq \emptyset$ because $I \setminus X_0 \notin F$. Furthermore, if $Y \in F$ then $Y \cap X_0 \neq \emptyset$ because if $Y \cap X_0 = \emptyset$ then $Y \subseteq I \setminus X_0$ and this would imply that $I \setminus X_0 \in F$ (by filter property (3)). So, if $X \in F'$ then either $\emptyset \neq X_0 \subseteq X$ or there is a $Y \in F$ with $\emptyset \neq Y \cap X_0 \subseteq X$; in either case, $X \neq \emptyset$. It easily follows from the definition of F' that $I \in F'$.
(2) Suppose $A, B \in F'$. By checking cases one can verify that $A \cap B \in F'$. For example, if $X_0 \subseteq A$ and there is a $Y \in F$ with $Y \cap X_0 \subseteq B$ then $Y \cap X_0 \subseteq A \cap B$ and hence $A \cap B \in F'$.
(3) Suppose $A \in F'$ and $A \subseteq B \subseteq I$. Then either $X_0 \subseteq A \subseteq B$ or there is a $Y \in F$ with $Y \cap X_0 \subseteq A \subseteq B$. In either case, $B \in F'$.

□

Proposition 2. *Every set with the finite intersection property can be extended to an ultrafilter.*

Proof. Suppose E is a collection of subsets of a set I and that E has the finite intersection property. Let $F_0 \supseteq E$ be the filter over I extending E obtained as in Exercise 3 above. Let $\mathcal{F} = \{F \mid F \text{ is a filter on } I \text{ and } F_0 \subseteq F\}$. Consider the partial order $(\mathcal{F}, \subseteq)$. We will use Zorn's Lemma to show that there is a maximal filter $U \in \mathcal{F}$, from which it will follow that $F_0 \subseteq U$ and U is an ultrafilter extending F_0 by Lemma 1 above.

In order to apply Zorn's Lemma to the partial order $(\mathcal{F}, \subseteq)$, we need to show that every $\subseteq$-increasing chain of filters over I has an upper bound in $(\mathcal{F}, \subseteq)$. Suppose $\mathcal{C} = \{F_j \mid j \in J\}$ is a chain of filters over the set I—where $(J, \leq_J)$ is some linearly ordered set. So $j \leq j'$ implies $F_j \subseteq F_{j'}$. Let $F = \bigcup_{j \in J} F_j$. Then clearly $F_j \subseteq F$ for every F_j in the chain. In order to show that F is an upper bound of the chain in the partial order $(\mathcal{F}, \subseteq)$, we need to show that $F \in \mathcal{F}$, or in other words, that F is a filter on I and $F_0 \subseteq F$. It is clear that $F_0 \subseteq F$ so it will suffice to check the three filter properties:

(1) $\emptyset \notin F$ because $\emptyset \notin F_j$ for every $j \in J$. Clearly $I \in F$.
(2) Suppose $A, B \in F = \bigcup_{j \in J} F_j$. Then there are fixed $j, j' \in J$ so that $A \in F_j$ and $B \in F_{j'}$. Since $\mathcal{C}$ forms a chain, it follows that either $F_j \subseteq F_{j'}$ or that $F_{j'} \subseteq F_j$. Without loss of generality, assume $F_j \subseteq F_{j'}$. Then $A, B \in F_{j'}$ and hence $A \cap B \in F_{j'} \subseteq F$.
(3) Suppose $A \in F$ and $A \subseteq B \subseteq I$. Then $A \in F_j$ for some $j \in J$. Since F_j is a filter it follows that $B \in F_j \subseteq F$.

Now, since every increasing chain in $(\mathcal{F}, \subseteq)$ has an upper bound, it follows from Zorn's Lemma that in the partial ordering $(\mathcal{F}, \subseteq)$, there is a maximal filter U extending F_0. By Lemma 1, U is an ultrafilter.

□

Exercise 4. If F is the Frechét filter over a set X and U is an ultrafilter with $F \subseteq U$ then U is not principle.

5. Ultraproducts and Łoś' Theorem

5.1. **Introduction to Direct Products.**

Consider the first-order structure $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot, \leq \rangle$. In this standard structure, the symbols $0, +, \cdot, \leq$ are interpreted in the usual way: $0^{\mathcal{N}} = 0$, $1^{\mathcal{N}} = 1$, $a +^{N} b = a + b$, $a \cdot^{\mathcal{N}} b = ab$, $a \leq^{\mathcal{N}} b \iff a \leq b$.

There is a natural way of interpreting the symbols $0, 1, +, \cdot, \leq$ within the cartesian product

$$\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n, m \in \mathbb{N}\}.$$

Let us define a new first order structure $\mathcal{M}_2 = \langle \mathbb{N} \times \mathbb{N}, 0^{\mathcal{M}}, 1^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}, \leq^{\mathcal{M}} \rangle$ as follows. Suppose $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, then

- $0^{\mathcal{M}_2} = (0, 0) = (0^{\mathcal{N}}, 0^{\mathcal{N}})$
- $1^{\mathcal{M}_2} = (1, 1) = (1^{\mathcal{N}}, 1^{\mathcal{N}})$
- $(a, b) +^{\mathcal{M}_2} (c, d) = (a + c, b + d) = (a +^{\mathcal{N}} c, b +^{\mathcal{N}} d)$
- $(a, b) \cdot^{\mathcal{M}_2} (c, d) = (ac, bd) = (a \cdot^{\mathcal{N}} c, b \cdot^{\mathcal{N}} d)$
- $(a, b) \leq^{\mathcal{M}_2} (c, d) \iff a \leq^{\mathcal{N}} c$ and $b \leq^{\mathcal{N}} d$

How are the structures $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot, \leq \rangle$ and

$$\mathcal{M}_2 = \langle \mathbb{N} \times \mathbb{N}, 0^{\mathcal{M}_2}, 1^{\mathcal{M}_2}, +^{\mathcal{M}_2}, \cdot^{\mathcal{M}_2}, \leq^{\mathcal{M}_2} \rangle$$

related? Do the structures satisfy the same first-order formulas?

Exercise 5. (a) Check that $\mathcal{M}_2$ satisfies the following sentence, but of course $\mathcal{N}$ does not.

$$\mathcal{M}_2 \models \exists x \exists y (x \neq 0 \wedge y \neq 0 \wedge x \cdot y = 0)$$

(b) Find a first-order sentence involving only $\leq$ that is true in $\mathcal{M}_2$ but false in $\mathcal{N}$. (Hint: For example, $\mathcal{M}_2$ consider the elements that are less than or equal to $1^{\mathcal{M}_2} = (1, 1)$.)

Remark 2. For $n \in \mathbb{N}$ we will use the symbol $\underline{n}$ as an abbreviation for the term $\underbrace{1 + \cdots + 1}_{n\text{-times}}$ in the language of arithmetic.

Exercise 6. Show that the structure $\mathcal{M}_2$ <u>does not</u> have "infinite" elements in the sense that for every element $(a, b) \in \mathbb{N} \times \mathbb{N}$ of the domain of $\mathcal{M}_2$ there is a natural number $k \in \mathbb{N}$ such that $\mathcal{M}_2 \models (a, b) \leq \underline{k}$ where $\underline{k}^{\mathcal{M}_2} = (\underbrace{1 + \cdots + 1}_{n\text{-times}})^{M_2} = (k, k)$.

We can also define a structures $\mathcal{M}_5$ on $\mathbb{N}^5 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathcal{M}_{\mathbb{N}}$ on $\mathbb{N}^{\mathbb{N}} = \{f \mid f \text{ is a function } f : \mathbb{N} \to \mathbb{N}\}$. One can just define the structure $\mathcal{M}_5$ coordinate-wise as we did above for $\mathcal{M}_2$. For $\mathcal{M}_{\mathbb{N}}$ one can also use a coordinate-wise interpretation by viewing the functions $f \in \mathbb{N}^{\mathbb{N}}$ as infinite sequences of natural numbers $f = (f(0), f(1), f(2), \ldots)$. A similar technique allows one to define a natural structure in the language of arithmetic on the set

$$\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\cdots$$

or even

$$\cdots\times\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\cdots.$$

Also, notice that there is nothing special in the above discussion about the language. This can all be carried our in any first-order language L.

5.2. **Direct Products and Filters.**

The next definition is general enough to handle all of the cases discussed in the previous section.

Definition 11. Suppose I is a nonempty index set and $\{\mathcal{M}_i \mid i\in I\}$ is a set of L-structures. The **direct product** of $\{\mathcal{M}_i \mid i\in I\}$ is an L structure $\mathcal{M}_I$ with domain

$$M_I := \prod_{i\in I} M_i = \{g \mid g \text{ is a function with } \mathrm{dom}(g)=I \text{ and } g(i)\in M_i\}$$

such that

- if c is a constant symbol from L then $c^{\mathcal{M}_I}\in M$ is a function with domain I such that $c^{\mathcal{M}_I}(i)=c^{\mathcal{M}_i}\in M_i$. (This defines a function g with domain I such that $g(i)=c^{\mathcal{M}_i}\in M_i$.)
- if f is an n-ary function symbol from L then we define the interpretation of f in $\mathcal{M}_I$ to be the function $f^{\mathcal{M}_I}: M_I^n\to M_I$ defined as follows. Given $(g_1,\ldots,g_n)\in M_I$, $f^{\mathcal{M}_I}(g_1,\ldots,g_n)$ will be an element of M_I. We define the function $f^{\mathcal{M}_I}(g_1,\ldots,g_n)$ with domain I by $f^{\mathcal{M}_I}(g_1,\ldots,g_n)(i)=f^{\mathcal{M}_i}(g_1(i),\ldots,g_n(i))$ for $i\in I$.
- if R is a relation symbol from L then $R^{\mathcal{M}_I}(g_1,\ldots,g_n)$ if and only if $R^{\mathcal{M}_i}(g_1(i),\ldots,g_n(i))$ for each $i\in I$.

Exercise 7. Let $\mathcal{M}_{\mathbb{N}}$ denote the direct product of $\mathbb{N}$-copies of the standard structure $\mathcal{N}$ in the language of arithmetic. In other words, $\mathcal{M}_{\mathbb{N}}$ is the direct product of the collection $\{\mathcal{M}_i : i\in\mathbb{N}\}$ where $\mathcal{M}_i=\mathcal{N}$ for every $i\in\mathbb{N}$. Show that this structure

$$\mathcal{M}_{\mathbb{N}}=\langle \mathbb{N}^{\mathbb{N}}, 0^{\mathcal{M}_{\mathbb{N}}}, 1^{\mathcal{M}_{\mathbb{N}}}, +^{\mathcal{M}_{\mathbb{N}}}, \cdot^{\mathcal{M}_{\mathbb{N}}}, \leq^{\mathcal{M}_{\mathbb{N}}}\rangle$$

<u>does have</u> an "infinite" element in the sense that there is an element $a\in\mathbb{N}^{\mathbb{N}}$ of the domain of the structure such that for every natural number $k\in\mathbb{N}$ we have

$$\mathcal{M}_{\mathbb{N}}\models \neg(a\leq \underline{k})\wedge\neg(\underline{k}\leq a)$$

(Hint: Consider $a=\mathrm{id}:\mathbb{N}\to\mathbb{N}$ the identity function, meaning $\mathrm{id}(n)=n$ for every $n\in\mathbb{N}$.)

Before defining the general notion of "ultraproduct" of L-structures, let me first discuss an example. Let $\mathcal{M}_{\mathbb{N}} = \langle \mathbb{N}^{\mathbb{N}}, 0^{\mathcal{M}_{\mathbb{N}}}, 1^{\mathcal{M}_{\mathbb{N}}}, +^{\mathcal{M}_{\mathbb{N}}}, \cdot^{\mathcal{M}_{\mathbb{N}}}, \leq^{\mathcal{M}_{\mathbb{N}}} \rangle$, where for example, $0^{\mathcal{M}_{\mathbb{N}}} = \langle 0, 0, 0, \ldots \rangle$ and if $a, b \in \mathbb{N}^{\mathbb{N}}$ then

$$\begin{aligned} a +^{\mathcal{M}_{\mathbb{N}}} b &= \langle a(0) + b(0), a(1) + b(1), a(2) + b(2), \ldots \rangle \\ &= \langle a(n) + b(n) \mid n \in \mathbb{N} \rangle \end{aligned}$$

Similarly, $a \leq^{\mathcal{M}_{\mathbb{N}}} b$ if and only if for every $n \in \mathbb{N}$ we have $a(n) \leq b(n)$.

Let F be the tail filter on $\mathbb{N}$ discussed above in Example 12. For two functions $a, b \in \mathbb{N}^{\mathbb{N}}$ define

$$a \sim_F b \iff \{n \in \mathbb{N} \mid a(n) = b(n)\} \in F.$$

It is easy to see that $\sim_F$ is an equivalence relation on $\mathbb{N}^{\mathbb{N}}$. Notice that $a \sim_F b$ if and only if a and b are eventually equal as functions on $\mathbb{N}$—i.e., there is a k such that $[k, \infty) \subseteq \{n \in \mathbb{N} \mid a(n) = b(n)\}$. In other words, we are identifying functions which are equal on a big set. Let $[a]_F = \{b \in \mathbb{N}^{\mathbb{N}} \mid b \sim_F a\}$ be the equivalence class of a. Let $\mathbb{N}^{\mathbb{N}}/F = \{[a]_F \mid a \in \mathbb{N}^{\mathbb{N}}\}$ be the collection of equivalence classes.

Let us define a new structure $\mathcal{M}_{\mathbb{N}}/F$ in the language containing the symbols $0, 1, +, \cdot, \leq$. The domain of $\mathcal{M}_{\mathbb{N}}/F$ is the collection of equivalence classes $\mathbb{N}^{\mathbb{N}}/F$ where we interpret the symbols $0, 1, +, \cdot, \leq$ as follows

- $0^{\mathcal{M}_{\mathbb{N}}/F} = [(0, 0, 0, \ldots)]_F$ and $1^{\mathcal{M}_{\mathcal{N}}/F} = [\langle 1, 1, 1, \ldots \rangle]_F$
- $[a]_F +^{\mathcal{M}_{\mathbb{N}}/F} [b]_F = [\langle a(0) + b(0), a(1) + b(1), \ldots \rangle]_F$ (add the representative functions component-wise and then take the equivalence class)
- $[a]_F \cdot^{\mathcal{M}_{\mathbb{N}}/F} [b]_F = [\langle a(0) \cdot b(0), a(1) \cdot b(1), \ldots \rangle]_F$
- $[a]_F \leq^{\mathcal{M}_{\mathbb{N}}/F} [b]_F$ if and only if $\{n \in \mathcal{N} \mid a(n) \leq b(n)\} \in F$

Notice that in particular, we are saying that $[a]_F \leq^{\mathcal{M}_{\mathbb{N}}/F} [b]_F$ if and only if the set on which the components of a are less or equal to the components of b is big according to the filter F. Also notice that, for example, the above definition of $[a]_F +^{\mathcal{M}_{\mathcal{N}}/F} [b]_F$ does not depend on the choice of representatives for the equivalence classes: Suppose $a \sim_F c$ and $b \sim_F d$ then we will show that $[a]_F +^{\mathcal{M}_{\mathbb{N}}/F} [b]_F = [c]_F +^{\mathcal{M}_{\mathbb{N}}/F} [d]_F$. The set $\{n \in \mathbb{N} \mid a(n) + b(n) = c(n) + d(n)\}$ is in the filter F because

$$\{n \mid a(n) = c(n)\} \cap \{n \mid b(n) = d(n)\} \subseteq \{n \in \mathbb{N} \mid a(n) + b(n) = c(n) + d(n)\}$$

and each of the sets $\{n \mid a(n) = c(n)\}$ and $\{n \mid b(n) = d(n)\}$ are in the filter (since $a \sim_F c$ and $b \sim_F d$). This shows that

$$\begin{aligned} [a]_F +^{\mathcal{M}_{\mathbb{N}}/F} [b]_F &= [\langle a(n) + b(n) \mid n \in \mathbb{N} \rangle]_F \\ &= [\langle c(n) + d(n) \mid n \in \mathbb{N} \rangle]_F \\ &= [c]_F +^{\mathcal{M}_{\mathbb{N}}/F} [d]_F \end{aligned}$$

This defines a structure

$$\mathcal{M}_{\mathbb{N}}/F = \langle \mathbb{N}^{\mathbb{N}}/F, 0^{\mathcal{M}_{\mathbb{N}}/F}, 1^{\mathcal{M}_{\mathbb{N}}/F}, +^{\mathcal{M}_{\mathbb{N}}/F}, \cdot^{\mathcal{M}_{\mathbb{N}}/F}, \leq^{\mathcal{M}_{\mathbb{N}}/F} \rangle$$

Fact 1. There is an element $[a]_F \in \mathcal{M}_{\mathbb{N}}/F$ such that for every natural number $k \in \mathbb{N}$ we have

$$\mathcal{M}_{\mathbb{N}}/F \models \underline{k} \leq [a]_F.$$

Note that this element $[a]_F$ is an "infinite" element in a strong sense: it is $\geq$ every "natural number" $\underline{k}^{\mathcal{M}_{\mathbb{N}}/F}$. Whereas, in Exercise 7 we saw that $\mathcal{M}_{\mathbb{N}}$ has an "infinite" element in the weaker sense that the element is $\leq$ any natural number $\underline{k}^{\mathcal{M}_{\mathbb{N}}}$.

Proof. Show that $[\mathrm{id}]_F$ is the desired element where $\mathrm{id} : \mathbb{N} \to \mathbb{N}$ is the identity function defined by $\mathrm{id}(n) = n$. □

Fact 2. There are sentences that are true in $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot, \leq \rangle$ but false in $\mathcal{M}_{\mathbb{N}}/F$. In particular,

$$\mathcal{N} \models \forall x \forall y (x \cdot y = 0 \implies (x = 0 \vee y = 0))$$

whereas

$$\mathcal{M}_{\mathbb{N}}/F \models \exists x \exists y (x \neq 0 \wedge y \neq 0 \wedge x \cdot y = 0).$$

Remark 3. In what follows, we will show that if we use an "ultrafilter" U instead of just a filter F to build $\mathcal{M}_{\mathbb{N}}/U$, then a first-order sentence will be true in $\mathcal{M}_{\mathbb{N}}/U$ if and only if it is true in $\mathcal{N}$.

5.3. **Ultraproducts.**

Let $\mathcal{M}_I = \prod_{i \in I} \mathcal{M}_i$ be the direct product of the collection of L-structures $\{\mathcal{M}_i \mid i \in I\}$, as defined above. Let U be an ultrafilter on I. Define a relation $\sim_U$ on M as follows. If $a, b \in M_I$ (so a and b are functions with domain I as above) then

$$a \sim_U b \text{ if and only if } \{i \in I \mid a(i) = b(i)\} \in U.$$

In other words, $a \sim_U b$ if and only if a and b are equal on a "big" set (a set in U).

Fact 3. $\sim_U$ is an equivalence relation on $M_I = \prod_{i \in I} M_i$.

Now, if $a \in M = \prod_{i \in I} M_i$, we let

$$[a]_U := \{b \in M \mid b \sim_U a\} = \{b \in M \mid \{i \in I \mid b(i) = a(i)\} \in U\}.$$

In other words, the functions a and b are in the same equivalence class if they are equal on a big set.

Definition 12. The **ultraproduct** $\mathcal{M}_I/U = \prod_{i \in I} M_i/U$ of a collection $\{\mathcal{M}_i \mid i \in I\}$ of (nonempty) L-structures is the L-structure defined as follows

- The domain of $\mathcal{M}_I/U$ is the set $M_I/U := \{[a]_U \mid a \in M = \prod_{i \in I} M_i\}$.
- Given a constant symbol c from L, set $c^{\mathcal{M}_I/U} = [\langle c_i^{M_i} \mid i \in I \rangle]_U$ (in other words, $c^{\mathcal{M}_I/U} = [c^{\mathcal{M}_I}]_U$).

- Given an n-ary function symbol f from L, we define the interpretation of f in $\mathcal{M}_I/U$ to be a function $f^{\mathcal{M}_I/U} : (M_I/U)^n \to \mathcal{M}_I/U$ defined by $f^{\mathcal{M}_I/U}([a_1]_U, \ldots, [a_n]_U) = [\langle f^{\mathcal{M}_i}(a_1(i), \ldots, a_n(i)) \mid i \in I\rangle]_U$ (in other words, $f^{\mathcal{M}_I/U}([a_1]_U, \ldots, [a_n]_U) = [f^{\mathcal{M}_I}(a_1, \ldots, a_n)]_U$).
- Given an n-place relation symbol R from L and $[a_1]_U, \ldots, [a_n]_U \in \mathcal{M}_I/U$, then $R^{\mathcal{M}_I/U}([a_1]_U, \ldots, [a_n]_U)$ if and only if
$$\{i \in I \mid R^{\mathcal{M}_i}(a_1(i), \ldots, a_n(i))\} \in U.$$

Exercise 8. Show that the above definition of ultraproduct does not depend on the choice of representatives of equivalence classes. Thus, the ultraproduct $\mathcal{M}_I/U = \prod_{i\in I} \mathcal{M}_i/U$ is well defined.

Definition 13. Suppose $\mathcal{M}_I/U$ is the ultrapower of the L-structures $\{\mathcal{M}_i \mid i \in I\}$ by an ultrafilter U over the set I. Suppose $a_1, \ldots, a_n \in M_I = \prod_{i\in I} M_i$, so each of $a_1, \ldots, a_n$ is a function with domain I such that $a_j(i) \in M_i$ for $1 \leq j \leq n$. Suppose φ is an L-formula with n free variables. We define
$$\|\varphi[a_1, \ldots, a_n]\| := \{i \in I \mid \mathcal{M}_i \models \varphi(a_1(i), \ldots, a_n(i))\}.$$

Exercise 9. Use induction on complexity of terms to prove that if t is an L-term and $(a_1, \ldots, a_n) \in \mathcal{M}_I^n = \left(\prod_{i\in I} \mathcal{M}_i\right)^n$ is a tuple then
$$t^{\mathcal{M}_I/U} \underbrace{[[a_1]_U, \ldots, [a_n]_U]}_{\text{variable assignment}} = [\langle t^{\mathcal{M}_i}[a_1(i), \ldots, a_n(i)] : i \in I\rangle]_U. \tag{5.1}$$

Theorem 1 (Łoś' Theorem)**.** *Suppose that I is a nonempty set, $\{\mathcal{M}_i \mid i \in I\}$ is a collection of L-structures, $\mathcal{M}_I = \prod_{i\in I} \mathcal{M}_i$ is their direct product, and U is an ultrafilter on I. Then, for every L-formula φ with n free variables and every n-tuple $(a_1, \ldots, a_n) \in \mathcal{M}_I^n$ we have*
$$\mathcal{M}_I/U \models \varphi([a_1]_U, \ldots, [a_n]_U) \iff \|\varphi[a_1, \ldots, a_n]\| \in U$$

Proof. The proof is by induction on the complexity of formulas.

We now begin to prove Theorem 1 by induction on complexity of formulas. Suppose φ is a term equation $t_1 = t_2$ and $(a_1, \ldots, a_n) \in M_I^n$ is a matching tuple. We have

$$\begin{aligned}
&\mathcal{M}_I/U \models t_1 = t_2\ [[a_1]_U, \ldots, [a_n]_U] \\
&\iff t_1^{\mathcal{M}_I/U}[[a_1]_U, \ldots, [a_n]_U] = t_2^{\mathcal{M}_I/U}[[a_1]_U, \ldots, [a_n]_U] \\
&\iff [\langle t_1^{\mathcal{M}_i}[a_1(i), \ldots, a_n(i)] : i \in I\rangle]_U = [\langle t_2^{\mathcal{M}_i}[a_1(i), \ldots, a_n(i)] : i \in I\rangle]_U \\
&\qquad\qquad \text{(by Exercise 9)} \\
&\iff \{i \in I : t_1^{\mathcal{M}_i}[a_1(i), \ldots, a_n(i)] = t_2^{\mathcal{M}_i}[a_1(i), \ldots, a_n(i)]\} \in U \\
&\qquad\qquad \text{(by definition of } \sim_U) \\
&\iff \{i \in I : \mathcal{M}_i \models t_1 = t_2\ [a_1(i), \ldots, a_n(i)]\} \in U \\
&\iff \|t_1 = t_2[a_1, \ldots, a_n]\| \in U
\end{aligned}$$

Now suppose φ is a relational formula $R(t_1,\dots,t_\ell)$ with matching tuple $(a_1,\dots,a_n)\in M_I^n$. Then

$$\begin{aligned}
&\mathcal{M}_I/U \models R(t_1,\dots,t_\ell)\ [[a_1]_U,\dots,[a_n]_U]\\
&\iff (t_1^{\mathcal{M}_I/U}[[a_1]_U,\dots,[a_n]_U],\dots,t_\ell^{\mathcal{M}_I/U}[[a_1]_U,\dots,[a_n]_U])\in R^{\mathcal{M}_I/U}\\
&\iff ([\langle t_1^{\mathcal{M}_i}[a_1(i),\dots,a_n(i)] : i\in I\rangle]_U,\dots,[\langle t_\ell^{\mathcal{M}_i}[a_1(i),\dots,a_n(i)] : i\in I\rangle]_U)\in R^{\mathcal{M}_I/U}\\
&\iff \{i\in I : (t_1^{\mathcal{M}_i}[a_1(i),\dots,a_n(i)],\dots,t_\ell^{\mathcal{M}_i}[a_1(i),\dots,a_n(i)])\in R^{\mathcal{M}_i}\}\in U\\
&\iff \{i\in I : \mathcal{M}_i\models R(t_1,\dots,t_\ell)\ [a_1(i),\dots,a_n(i)]\}\in U\\
&\iff \|R(t_1,\dots,t_\ell)[a_1,\dots,a_n]\|\in U
\end{aligned}$$

So, we have shown that the theorem holds for atomic formulas. Now suppose the theorem is true for formulas φ_1 and φ_2, we need to show it's true for $\varphi_1\wedge\varphi_2$. Let $(a_1,\dots,a_n)\in\mathcal{M}_I^n$ be a tuple matching the number of free variables of $\varphi_1\wedge\varphi_2$. We have

$$\begin{aligned}
\mathcal{M}_I/U &\models \varphi_1\wedge\varphi_2[[a_1]_U,\dots,[a_n]_U]\\
&\iff \mathcal{M}_I/U\models\varphi_1[[a_1]_U,\dots,[a_n]_U] \text{ and } \mathcal{M}_I/U\models\varphi_2[[a_1]_U,\dots,[a_n]_U]\\
&\qquad\text{(by induction)}\\
&\iff \|\varphi_1[a_1,\dots,a_n]\|\in U \text{ and } \|\varphi_2[a_1,\dots,a_n]\|\in U\\
&\iff \|\varphi_1[a_1,\dots,a_n]\|\cap\|\varphi_2[a_1,\dots,a_n]\|\in U\\
&\qquad\text{(because } U \text{ is a filter)}\\
&\iff \|\varphi_1\wedge\varphi_2[a_1,\dots,a_n]\|\in U
\end{aligned}$$

Now suppose the theorem is true for a formula φ. We need to show it's true for $\neg\varphi$. (Note: this is where we are using that the filter U is an *ultra*filter.) Suppose $(a_1,\dots,a_n)\in\mathcal{M}_I^n$ is a tuple matching the number of free variables of φ. We have

$$\begin{aligned}
&\mathcal{M}_I/U\models\neg\varphi[[a_1]_U,\dots,[a_n]_U]\\
&\iff \text{it is not the case that } \mathcal{M}_I/U\models\varphi[[a_1]_U,\dots,[a_n]_U]\\
&\iff \|\varphi[a_1,\dots,a_n]\|\notin U\\
&\iff I\setminus\|\varphi[a_1,\dots,a_n]\|\in U \qquad \text{(because } U \text{ is an } \textit{ultra}\text{filter)}\\
&\iff \|\neg\varphi[a_1,\dots,a_n]\|\in U
\end{aligned}$$

Now suppose the theorem is true for $\varphi(x,y_1\dots,y_n)$. We must show the theorem is true for $\exists x\ \varphi(x,y_1,\dots,y_n)$. Suppose $(a_1,\dots,a_n)\in M_I^n$. We have

$$\begin{aligned}
\mathcal{M}_I/U \models \exists x\varphi[[a_1]_U, \dots, [a_n]_U] \\
&\iff \text{there is a } [b]_U \in M_I/U \text{ with} \\
&\qquad \mathcal{M}_I/U \models \varphi[[b]_U, [a_1]_U, \dots, [a_n]_U] \\
&\iff \text{there is a } [b]_U \in M_I/U \text{ with} \\
&\qquad \|\varphi[b, a_1, \dots, a_n]\| \in U \qquad \text{(by induction)} \\
&\iff \|\exists x\varphi[a_1, \dots, a_n]\| \in U
\end{aligned}$$

□

Exercise 10. Explain the last equivalence in the previous proof.

Definition 14. Suppose I is an infinite set, U is an ultrafilter on I, and for each $i \in I$ we have $\mathcal{M}_i = \mathcal{M}$. The **ultrapower** of $\mathcal{M}$ by U is defined to be the L-structure $\prod \mathcal{M}_i/U = \mathcal{M}^I/U$. The domain of this structure is the set of functions

$$M^I = \{f : \text{such that } f \text{ is a function from } I \text{ to } M\}.$$

- If c is a constant symbol in L we define $c^{\mathcal{M}^I/U} = [\langle c^{\mathcal{M}_i} : i \in I\rangle]_U = [\langle c^{\mathcal{M}} : i \in I\rangle]_U$ to be the equivalence class of the constant function $i \mapsto c^{\mathcal{M}}$.
- If f is an n-ary constant symbol and $[a_1]_U, \dots, [a_n]_U \in M^I/U$, we define $f^{\mathcal{M}^I/U}([a_1]_U, \dots, [a_n]_U) = [\langle f^{\mathcal{M}}(a_1(i), \dots, a_n(i)) : i \in I\rangle]_U$.
- If R is an n-ary relation symbol and $[a_1]_U, \dots, [a_n]_U \in M^I/U$ define

$$([a_1]_U, \dots, [a_n]_U) \in R^{\mathcal{M}^I/U} \iff \{i \in I : (a_1(i), \dots, a_n(i)) \in R^{\mathcal{M}}\} \in U$$

Example 14. Suppose U is a nonprinciple ultrafilter in $\mathcal{N}$ (extend the tail filter). Let $\mathcal{N}$ be the standard model of arithmetic. By Łoś' theorem, if $\mathcal{N}^{\mathbb{N}}/U$ is the ultrapower of copies of $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \cdot, \leq\rangle$ then $\mathcal{M}^{\mathbb{N}}/U$ and $\mathcal{N}$ satisfy all of the same first order formulas, even though $\mathcal{M}_{\mathbb{N}}/U$ has "infinite elements" in a strong sense.

Exercise 11. Suppose $\mathcal{M}$ is an L-structure and U is an ultrafilter over some infinite set I. For each $a \in M$ define $c_a : I \to M$ to be the function with domain I with constant value a: $c_a(i) = a$ for all $i \in I$. Use Łos' Theorem to prove that the map $e : \mathcal{M} \to \mathcal{M}^I/U$ defined by $e(a) = [c_a]_U$ is an elementary embedding.

6. Proof of the Compactness Theorem Using Ultraproducts

Theorem 2 (The Compactness Theorem)**.** *A set of L-sentences Σ has a model if and only if every finite subset of Σ has a model.*

Proof. ($\longrightarrow$) This direction is trivial because if $M \models \Sigma$ then M satisfies every finite subset of Σ.

($\longleftarrow$) Suppose every finite subset of Σ has a model. Let I be the collection of all finite subsets of Σ. For each $i \in I$ let $\mathcal{M}_i \models i$. For each $i \in I$ let $i^* := \{j \in I \mid i \subseteq j\}$. It is easy to see that the set $I^* := \{i^* \mid i \in I\}$ has the finite intersection property because if $i_0^*, i_1^* \in I^*$ then $i_0^* \cap i_1^* = (i_0 \cup i_1)^* \neq \emptyset$. By Proposition 2, it follows that there is an ultrafilter U over I with $I^* \subseteq U$.

Let $\mathcal{M}_I := \prod_{i\in I} \mathcal{M}_i$ be the product of the L-structures $\{\mathcal{M}_i \mid i \in I\}$ and let $\mathcal{M}_I/U := \prod_{i\in I} \mathcal{M}_i/U$ denote the ultraproduct associated to U. We will show that $\mathcal{M}_I/U \models \Sigma$. Choose a sentence $\varphi \in \Sigma$. Then $\{\varphi\} \in I$ and $\mathcal{M}_i \models \varphi$ for all i with $\varphi \in i$. In other words

$$\{\varphi\}^* = \{i \in I \mid \varphi \in i\} \subseteq \{i \in I \mid \mathcal{M}_i \models \varphi\}.$$

Since $\{\varphi\}^* \in I^*$ and U was chosen so that $I^* \subseteq U$, it follows that $\{\varphi\}^* \in U$. Then filter axiom (3) implies that $\{i \in I \mid \mathcal{M}_i \models \varphi\} \in U$. Finally, Łoś' Theorem implies $\mathcal{M}_I/U \models \varphi$. Since φ was a arbitrary element of Σ, we see that $M_U \models \Sigma$, as desired. □

The next important theorem follows from the compactness theorem and has many interesting applications. For example, it immediately implies that there are structures which satisfy all of the same first-order statements as the natural numbers $\langle \mathbb{N}, 0, 1, +, \cdot\ \leq \rangle$ but which are uncountable, and indeed, of any cardinality.

Theorem 3 (The Löwenheim-Skolem Theorem)**.** *If a collection of L-formulas Σ has arbitrarily large finite models or an infinite model, then Σ has models of arbitrarily large cardinality.*

Proof. For any given set C of *new constant symbols* (meaning constant symbols not already in L), we are going to find a model of Σ of cardinality at least the cardinality of C.

Consider the following set of $L(C)$-sentences.

$$\Sigma_C = \Sigma \cup \{c \neq c' \mid c, c' \in C \text{ and } c \neq c'\}$$

Since Σ has arbitrarily large finite models or an infinite model, every finite subset $\Sigma_0 \subseteq \Sigma$ has a model, because in any sufficiently large structure we can find pairwise distinct interpretations for the finitely many c occuring in Σ_0. Thus the compactness theorem yields a model of Σ_C. Its L-reduct is then a model of Σ of cardinality at least $|C|$. □

7. Types

Given an L-structure $\mathcal{M}$ and a subset $A \subseteq M$, we define an expanded language $L_A = L \cup A$ where we view the elements of A as new constant symbols. We define a new structure $(\mathcal{M}, A)$ in the expanded language as follows. The new structure interprets the symbols of L in exactly the same way that $\mathcal{M}$ does: $c^{(\mathcal{M},A)} = c^{\mathcal{M}}$, $f^{(\mathcal{M},A)} = f^{\mathcal{M}}$ and $R^{(\mathcal{M},A)} = R^{\mathcal{M}}$. The new structure interprets the new constant symbols in the obvious way: if $a \in A$ is a new constant in the language L_A we define $a^{(\mathcal{M},A)} = a$. For example, using this notation, the elementary diagram of the L-structure $\mathcal{M}$ is precisely the theory of the L_M-structure $(\mathcal{M}, M)$:

$$\begin{aligned}\operatorname{diag}_{\mathrm{el}}(\mathcal{M}) &= \operatorname{Th}(\mathcal{M}, M) \\ &= \text{“the set of all } L_M\text{-sentences true in } (\mathcal{M}, M)\text{”}\end{aligned}$$

If $\varphi(x_1, \ldots, x_n)$ is an L-formula and $(a_1, \ldots, a_n)$ is a tuple from M, we can define and L_M sentence $\varphi(a_1, \ldots, a_n)$ by induction on the complexity of φ in the obvious way so that the following fact is true. For example if $\varphi(x_1, x_2)$ is the L-formula $x_1 = x_2$ then $\varphi(a_1, a_2)$ is the L_M sentence $a_1 = a_2$.

Fact 4. If $\mathcal{M}$ is an L-structure, $\varphi(\bar{x})$ is an L-formula where $\bar{x} = (x_1, \ldots, x_n)$, and $\bar{a} = (a_1, \ldots, a_n)$ is a matching tuple from M then

$$\underbrace{\mathcal{M} \models \varphi[a_1, \ldots, a_n]}_{\text{varphiable assignment}} \iff (\mathcal{M}, M) \models \underbrace{\varphi(a_1, \ldots, a_n)}_{\text{an } L_M\text{-formula}}.$$

Example 15. Suppose $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot, <)$ is the standard model in the language L of arithmetic. Then in the language $L_{\mathbb{N}} = L \cup \mathbb{N}$ we have constants for every element of the domain, and expressions of the form $(3+1 = 4) \wedge (5 < 17)$ are $L_{\mathbb{N}}$-sentences. We can also write an equivalent form of the above expression in the smaller language L: $(\underline{3} + \underline{1}) = \underline{4}) \wedge (\underline{5} < \underline{17})$.

Remark 4. In what follows, if $\mathcal{M}$ is an L-structure and $\bar{a} = (a_1, \ldots, a_n) \in M^n$ is a tuple matching an L-formula φ, when we write $\mathcal{M} \models \varphi(\bar{a})$ we will mean $(\mathcal{M}, M) \models \varphi(\bar{a})$ (or equivalently $\mathcal{M} \models \varphi(\bar{x})[\bar{a}]$).

Let $\mathcal{M}$ be an L-structure, suppose $A \subseteq M$ and let $\bar{b} = (b_1, \ldots, b_n) \in M^n$. The **complete type of $\bar{b}$ over** A (with respect to $\mathcal{M}$ in the variables $\bar{x} = (x_1, \ldots, x_n)$) is defined to be following collection of L_A-formulas.

$$\begin{aligned}\operatorname{tp}^{\mathcal{M}}(\bar{b}/A) &= \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in A^k \text{ for some } k \text{ and } \mathcal{M} \models \varphi(\bar{b}, \bar{a})\} \\ &= \{\varphi(\bar{x}) : \varphi(\bar{x}) \text{ is an } L_A\text{-formula and } \mathcal{M} \models \varphi(\bar{b})\}\end{aligned}$$

In other words, the complete type of $\bar{b}$ over A is the collection of all matching formulas with parameters from A that are true about $\bar{b}$ in $\mathcal{M}$. We say that a set of L_A-formulas $p(\bar{x})$ is a **complete type over** A (with respect to $\mathcal{M}$ in the variables $\bar{x}$) if it is the complete type of some tuple $\bar{b}$ over A with respect to some elementary extension of $\mathcal{M}$. In other words, $p(\bar{x})$ is a complete type over A if there is an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ and a tuple $\bar{b}$ from N such that $p(\bar{x}) = \operatorname{tp}^{\mathcal{N}}(\bar{b}/A)$.

A subset $\Phi(\bar{x}) \subseteq p(\bar{x})$ of a complete type over A is called a **type** over A (with respect to $\mathcal{M}$ in the variables $\bar{x} = (x_1, \ldots, x_n)$). We say that a type $\Phi(\bar{x})$ over A is **realized** by a tuple $\bar{b}$ in M if $\Phi(\bar{x}) \subseteq \operatorname{tp}^{\mathcal{M}}(\bar{b}/A)$; or, in other words, $\mathcal{M} \models \Phi(\bar{b})$ (which means $(\mathcal{M}, M) \models \varphi(\bar{b})$ for each $\varphi(\bar{x}) \in \Phi(\bar{x})$). A type $\Phi(\bar{x})$ is **finitely realized** in $\mathcal{M}$ if for every finite subset $\Psi(\bar{x}) \subseteq \Phi(\bar{x})$

there is a tuple $\bar{b}$ from $\mathcal{M}$ such that for every $\psi(\bar{x}) \in \Psi(\bar{x})$ we have $\mathcal{M} \models \psi(\bar{b})$.

Example 16.

(1) If $\mathcal{M}$ is an infinite L-structure then $\{x \neq a : a \in M\}$ is a 1-type of $\mathcal{M}$ that is realized in an elementary extension $\mathcal{N}$ of $\mathcal{M}$, precisely by those elements that are not in $\mathcal{M}$.
(2) $\{n < x : n \in \mathbb{N}\}$ is a 1-type over $\mathbb{N}$ of the standard model of arithmetic $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot, <)$. Note, here n really means n and not $\underline{n} = 1 + \cdots + 1$. It is the case that $\{\underline{n} < x : n \in \mathbb{N}\}$ is a type over $\emptyset$ of the standard model of arithmetic.
(3) Every Dedekind cut in can be viewed as a 1-type of $(\mathbb{Q}, <)$.

Theorem 4. *Let $\mathcal{M}$ be an L-structure, $A \subseteq M$ a set of parameters, $\Phi(x_1, \ldots, x_n)$ a set of formulas of L with parameters from A. Then, writing $\bar{x}$ for $(x_1, \ldots, x_n)$,*

(a) *$\Phi(\bar{x})$ is a type over A with respect to $\mathcal{M}$ if and only if Φ is finitely realized in $\mathcal{M}$.*
(b) *$\Phi(\bar{x})$ is a complete type over A with respect to $\mathcal{M}$ if and only if $\Phi(\bar{x})$ is a set of formulas of L with parameters from A, which is maximal with the property that it is finitely realized in $\mathcal{M}$.*

In particular, if Φ is finitely realized in $\mathcal{M}$, then it can be extended to a complete type over A with respect to $\mathcal{M}$.

Proof. (a) ($\longrightarrow$) Suppose $\Phi(\bar{x})$ is a type over A with respect to $\mathcal{M}$. Then by definition $\Phi(\bar{x})$ is a subset of the complete type of some matching tuple from an elementary extension of $\mathcal{M}$. So there is an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ and a matching tuple $\bar{b}$ from $\mathcal{N}$ with $\mathcal{N} \models \Phi(\bar{b})$. If Ψ is a finite subset of Φ then $\mathcal{N} \models \bigwedge \Psi(\bar{b})$ and hence $\mathcal{N} \models \exists \bar{x} \bigwedge \Psi(\bar{x})$. Since $\mathcal{M} \preccurlyeq \mathcal{N}$ we have $\mathcal{M} \models \exists \bar{x} \bigwedge \Psi(\bar{x})$, and thus there is a tuple $\bar{a}$ from $\mathcal{M}$ with $\mathcal{M} \models \Psi(\bar{a})$. Thus $\Psi(\bar{x})$ is finitely realized in $\mathcal{M}$. ($\longleftarrow$) Suppose $\Phi(\bar{x})$ is finitely realized in $\mathcal{M}$. We must show that $\Phi(\bar{x})$ is realized in some elementary extension of $\mathcal{M}$. Recall that $\text{diag}_{\text{el}}(\mathcal{M})$ is the collection of all L_M-formulas true in $\mathcal{M}$. Let $\bar{c} = (c_1, \ldots, c_n)$ be a tuple of new constants matching $\Phi(\bar{x})$. One can use the compactness theorem to prove that $\text{diag}_{\text{el}}(\mathcal{M}) \cup \Phi(\bar{c})$ is consistent. Thus there is an $L_M \cup \{c_1, \ldots, c_n\}$ model $\mathcal{N} \models \text{diag}_{\text{el}}(\mathcal{M}) \cup \Phi(\bar{c})$, and an elementary embedding $e : \mathcal{M} \to \mathcal{N} \upharpoonright L$. Without loss of generality we can assume $\mathcal{M} \preccurlyeq \mathcal{N} \upharpoonright L$. Let $\bar{b} = \bar{c}^{\mathcal{N}}$ and notice that $\mathcal{N} \models \Phi(\bar{b})$ since $\mathcal{N} \models \Phi(\bar{c})$. Thus Φ is realized in an elementary extension of $\mathcal{M}$, and so Φ is a type over A with respect to M.

(b) ($\longrightarrow$) Suppose $\Phi(\bar{x})$ is a complete type over A with respect to $\mathcal{M}$. This means that $\Phi(\bar{x})$ is the complete type some tuple $\bar{b}$ in some elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$; in other words, $\Phi(\bar{x}) = \text{tp}^{\mathcal{N}}(\bar{b}/A)$. Since every L_A-formula is either true or false about $\bar{b}$ in $\mathcal{N}$, it follows that for every L_A formula $\varphi(\bar{x})$ either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$. This implies that $\Phi(\bar{x})$ is a maximal type over A. ($\longleftarrow$) Suppose $\Phi(\bar{x})$ is a maximal type over A. It follows that $\Phi(\bar{x})$ is a realized by some tuple $\bar{b}$ is some elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$.

We have $\Phi(\bar{x}) \subseteq \mathrm{tp}^{\mathcal{N}}(\bar{b}/A)$, and since $\Phi(\bar{x})$ is a maximal type over A we must have $\Phi(\bar{x}) = \mathrm{tp}^{\mathcal{N}}(\bar{b}/A)$, thus Φ is a complete type over A. □

Lemma 2. *Suppose $A \subseteq M, N$ and $e : \mathcal{M} \to \mathcal{N}$ is an isomorphism fixing A (i.e. $e(a) = a$ for every $a \in A$). If Φ is a type of $\mathcal{M}$ over A, then Φ is also a type of $\mathcal{N}$ over A. Furthermore, if $\bar{a}$ is a matching tuple from M then $M \models \Phi(\bar{a})$ if and only if $\mathcal{N} \models \Phi(e[\bar{a}])$.*

Proof. Recall that isomorphisms are elementary embeddings.

Suppose Φ is a type of $\mathcal{M}$ over A. Then by the previous theorem, this means that Φ is finitely realized in $\mathcal{M}$. We must show that Φ is also finitely realized in $\mathcal{N}$. Suppose $\Psi(\bar{x}) \subseteq \Phi(\bar{x})$ is finite. Then there is a tuple $\bar{b}$ from M such that $\mathcal{M} \models \bigwedge \Psi(\bar{b})$. By the elementarily of e, it follows that $\mathcal{N} \models \bigwedge \Psi(e[\bar{b}])$.

The rest follows by elementarily of e.

□

8. Saturation of ultraproducts and the Keisler-Shelah Theorem

Definition 15. An L-structure $\mathcal{M}$ is **κ-saturated** if it realizes all of its 1-types over subsets $A \subseteq M$ with $|A| < \kappa$. The structure $\mathcal{M}$ is called **saturated** if it is $|M|$-saturated.

Example 17.

(1) Every finite structure is saturated (and even κ-saturated for all κ). This is simply because finite structures do not have proper elementary extensions.
(2) A countable structure $\mathcal{M}$ is saturated if and only if $\mathcal{M}$ realizes all of its 1-types having only finitely many parameters.
(3) If $\mathcal{M}$ is a countable structure then $\mathcal{M}$ is not ω_1-saturated.

Fact 5. Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $f : M \to N$. Then f is an elementary embedding if and only if $(\mathcal{M}, M) \equiv (\mathcal{N}, f[M])$. Note: here $(\mathcal{N}, f[M])$ is the L_M-structure in which if $a \in M$ then $a^{(\mathcal{N}, f[M])} = f(a)$.

Theorem 5. *Let $\mathcal{M}$ be a countably infinite saturated L-structure.*

(1) *(Universality) Every countable $\mathcal{N} \equiv \mathcal{M}$ can be elementarily embedded in $\mathcal{M}$.*
(2) *(Uniqueness) If $\mathcal{N} \equiv \mathcal{M}$ is countable and saturated, then $\mathcal{N} \cong \mathcal{M}$.*
(3) *(Homogeneity) Suppose A is a finite subset of $\mathcal{M}$ and $\bar{a}$ and $\bar{b}$ are tuples of the same length from M. Then $tp^{\mathcal{M}}(\bar{a}/A) = tp^{\mathcal{M}}(\bar{b}/A)$ if and only if there is a $\sigma \in \mathrm{Aut}_A(\mathcal{M})$ such that $\sigma[\bar{a}] = \bar{b}$.*

Proof. (Sketch) (1) Suppose $N = \{b_i : i \in \mathbb{N}\}$ is an enumeration of N. By induction on i we choose $a_i \in M$ such that $(\mathcal{N}, b_0, \ldots, b_n) \equiv (\mathcal{M}, a_1, \ldots, a_n)$ for all $n \in \mathbb{N}$. Then we define $e(b_i) = a_i$ to obtain the desired map. Suppose (as the inductive hypothesis) that $\bar{a} = (a_0, \ldots, a_{n-1})$ has already been chosen so that $(\mathcal{M}, \bar{a}) \equiv (\mathcal{N}, \bar{b})$ where $\bar{b} = (b_0, \ldots, b_{n-1})$.

Let $\Phi_n = \{\psi(x,\bar{b}) : \psi(x,\bar{b}) \in \mathrm{tp}^{\mathcal{N}}(b_n/\bar{b})\}$. This set is closed under finite conjunctions, as so is $\mathrm{tp}^{\mathcal{N}}(b_n/\bar{b})$. Furthermore, since $\mathcal{N} \models \exists x \psi(x,\bar{b})$ for all $\psi(x,\bar{b}) \in \mathrm{tp}^{N}(b_n/\bar{b})$, the induction hypothesis gives $\mathcal{M} \models \exists x \psi(x,\bar{a})$, thus Φ_n is a type by Theorem 4.

As $\mathcal{M}$ is saturated, Φ_n is realized in $\mathcal{M}$ by some a_n. Consequently $(\mathcal{M}, a_0, \ldots, a_n) \equiv (\mathcal{N}, b_0, \ldots, b_n)$, and we define $e(b_n) = a_n$.

By induction, this defines a map $e : N \to M$, which is an elementary embedding.

(2) Suppose $\mathcal{N}$ is also saturated and $N = \{b_i : i \in \mathbb{N}\}$. Use back-and-forth to inductively construct an isomorphism.

(3) Exercise.

□

Remark 5. Recall that by definition, an L-structure $\mathcal{M}$ is ω_1-saturated if and only if $\mathcal{M}$ realizes all of its 1-types over subsets $A \subseteq M$ with $|A| = \mathbb{N}$.

Theorem 6. *Suppose $I = \mathbb{N}$ and let U be a non principle ultrafilter over I. If L is a countable language and $\{\mathcal{M}_i : i \in I\}$ is a collection of L-structures, then the ultraproduct $\mathcal{M}_I/U = \prod_{i\in I} \mathcal{M}_i/U$ is ω_1-saturated.*

Proof. To show that the ultraproduct $\mathcal{M}_I/U$ is ω_1-saturated, we must show that every type over a countable subset of the domain is realized in the ultraproduct $\mathcal{M}_I/U$. By Theorem 4(a), this amounts to showing that if $\Phi(x)$ is a set of L_A-formulas for some countable $A \subseteq M_I/U$ and if $\Phi(x)$ is finitely realized in $\mathcal{M}_I/U$, then $\Phi(x)$ is realized in $\mathcal{M}_I/U$.

Since the language L_A is countable, we can let $\Phi(x) = \{\varphi_n(x) : n \in \mathbb{N}\}$ be an enumeration of $\Phi(x)$. Let us fix a countable descending sequence of elements of U

$$I = I_0 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

such that $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$ (for example we could take $I_n = [n, \infty) \cap \mathbb{N}$). Now for each $n \geq 1$ define

$$X_n := I_n \cap \{i \in I : \mathcal{M}_i \models \exists x(\varphi_1(x) \wedge \cdots \wedge \varphi_n(x))\}.$$

We have $X_n \supseteq X_{n+1}$ for each $n \geq 1$ and furthermore, $\bigcap_{n\geq 1} X_n = \emptyset$. Since $\Phi(x)$ is finitely realized in $\mathcal{M}_I/U$ we have $\{i \in I : \mathcal{M}_i \models \exists x(\varphi_1(x) \wedge \cdots \wedge \varphi_n(x))\} \in U$, and thus $X_n \in U$ for $n \geq 1$.

Since $\bigcap_{n\geq 1} X_n = \emptyset$, for each $i \in \bigcup_{n\geq 1} X_n$ there is a largest natural number $n(i)$ such that $i \in X_{n(i)}$. For $i \notin \bigcup_{n\geq 1} X_n$ define $n(i) = 0$. Now we define an element $f \in \prod_{i\in I} M_i$ as follows, with aim to show that $\mathcal{M}_I/U \models \Phi([f]_U)$. If $i \in \bigcup_{n\geq 1} X_n$ we choose $f(i) \in M_i$ such that $\mathcal{M}_i \models \varphi_1(f(i)) \wedge \cdots \wedge \varphi_{n(i)}(f(i))$. If $i \notin \bigcup_{n\geq 1} X_n$ we let $f(i)$ be any element of M_i. Since $\Phi(x) = \{\varphi_n(x) : n \in \mathbb{N}\}$, to show that $\mathcal{M}_I/U \models \Phi([f]_U)$ we suppose $\varphi_n(x) \in \Phi(x)$ and show that $X_n \subseteq \{i \in I : \mathcal{M}_i \models \varphi_n(f(i))\} \in U$. Suppose $i \in X_n$, then $n \leq n(i)$ and so $i \in X_{n(i)}$, which implies $\mathcal{M}_i \models \varphi_1(f(i)) \wedge \cdots \wedge \varphi_n(f(i))$, and hence $\mathcal{M}_i \models \varphi_n(f(i))$. Thus $\{i \in I : \mathcal{M}_i \models \varphi_n(f(i))\} \in U$ and by Łos' Theorem we have $\mathcal{M}_I/U \models \varphi_n([f]_U)$.

□

We will need some basic set-theoretic facts and terminology to continue our discussion of saturation.

If X and Y are sets then ${}^{Y}X$ denotes the collection of all functions from Y to X. If Y is a set then ${}^{Y}2$ denotes the set of all 0-1-valued functions with domain Y. Note that there is clearly a bijection from the powerset of Y, $P(Y)$, to the set ${}^{Y}2$: just map a subset of Y to its characteristic function.

Examples 1.

- ${}^{\mathbb{N}}2$ denotes the set of all sequences of 0's and 1's.
- Similarly, ${}^{\mathbb{N}\times\mathbb{N}}2$ denotes the set of all 0-1-valued functions with domain $\mathbb{N}\times\mathbb{N}$—each such function can be visualized as an infinite grid of 0's and 1's.
- The set ${}^{\mathbb{N}}({}^{\mathbb{N}}2)$ denotes the set of all sequences of sequences of 0's and 1's.

Fact 6. $|{}^{\mathbb{N}}({}^{\mathbb{N}}2)| = |{}^{\mathbb{N}\times\mathbb{N}}2| = |{}^{\mathbb{N}}2|$

Proof. For the first equality, there is a bijection $f : {}^{\mathbb{N}\times\mathbb{N}}2 \to {}^{\mathbb{N}}({}^{\mathbb{N}}2)$. Given an infinite grid of 0's and 1's, $G \in {}^{\mathbb{N}\times\mathbb{N}}2$, we define $f(G) = \langle \vec{a}_n : n \in \mathbb{N}\rangle$ to be a sequence of sequences of 0's and 1's by letting $\vec{a}_n$ be the n^{th}-column of the grid.

For the second equality, just use the fact that $|\mathbb{N}\times\mathbb{N}| = \mathbb{N}$ (the product of two countable sets is countable). □

We denote the least infinite cardinal by ω, and in fact $\omega = \mathbb{N}$. The least uncountable cardinal is denoted by ω_1 (there is no bijection from ω to ω_1). **The continuum hypothesis** (CH) states that $|P(\mathbb{N})| = \omega_1$, or equivalently $|{}^{\mathbb{N}}2| = \omega_1$.

Lemma 3. *If $\mathcal{M}$ and $\mathcal{N}$ are saturated models of the same cardinality then $\mathcal{M} \cong \mathcal{N}$. (Similar to Theorem 5(2).)*

Lemma 4. *If $(\mathcal{M}, \bar{a}) \equiv (\mathcal{N}, \bar{b})$ and $\bar{a}$ covers M and $\bar{b}$ covers N, then $\mathcal{M} \cong \mathcal{N}$.*

Theorem 7. *(Keisler-Shelah) Assume that $2^{\omega} = \omega_1$. Suppose L is a countable language and let $\mathcal{M}$ and $\mathcal{N}$ be L-structures of cardinality at most ω_1. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if there is some ultrafilter U over $\mathbb{N}$ such that $\mathcal{M}^{\mathbb{N}}/U \cong \mathcal{N}^{\mathbb{N}}/U$.*

Proof. ($\longleftarrow$) Suppose $\mathcal{M}^{\mathbb{N}}/U \cong \mathcal{N}^{\mathbb{N}}/U$, then by Łos' Theorem we have $\mathcal{M} \equiv \mathcal{M}^{\mathbb{N}}/U \cong \mathcal{N}^{\mathbb{N}}/U \equiv \mathcal{N}$.

($\longrightarrow$) Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures with $|M|, |N| \leq \omega_1$. Suppose $\mathcal{M} \equiv \mathcal{N}$ and let U be a nonprinciple ultrafilter over $\mathbb{N}$. By Theorem 6, the structures $\mathcal{M}^{\mathbb{N}}/U$ and $\mathcal{N}^{\mathbb{N}}/U$ are both ω_1-saturated. Since each equivalence class in $M^{\mathbb{N}}/U$ is represented by a function (or sequence) in the cartesian product

$\prod_{i\in\mathbb{N}} M$, we have

$$\left|M^{\mathbb{N}}/U\right| \leq \left|\prod_{i\in\mathbb{N}} M\right| \leq |{}^{\mathbb{N}}M| = |{}^{\mathbb{N}}({}^{\mathbb{N}}2)| = |{}^{\mathbb{N}\times\mathbb{N}}2| = |{}^{\mathbb{N}}2| = |P(\mathbb{N})| = \omega_1$$

Similarly, $|N^{\mathbb{N}}/U| \leq \omega_1$. Since both $\mathcal{M}^{\mathbb{N}}/U$ and $\mathcal{N}^{\mathbb{N}}/U$ are ω_1-saturated, neither can have cardinality ω, thus the domains of both ultrapowers must have cardinality precisely ω_1. By the uniqueness of saturated models in a given cardinality, e.g. Lemma 3, it follows that $\mathcal{M}^{\mathbb{N}}/U \cong \mathcal{N}^{\mathbb{N}}/U$.

□

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