

SUPERCOMPACTNESS AND THE CONTINUUM FUNCTION

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ABSTRACT. Given a cardinal κ that is λ -supercompact for some regular cardinal $\lambda \geq \kappa$ and assuming GCH, we show that one can force the continuum function to agree with any function $F : [\kappa, \lambda] \cap \text{REG} \rightarrow \text{CARD}$ satisfying $\forall \alpha, \beta \in \text{dom}(F) \alpha < \text{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leq F(\beta)$, while preserving the λ -supercompactness of κ from a hypothesis that is of the weakest possible consistency strength, namely, from the hypothesis that there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > F(\lambda)$. Our argument extends Woodin's technique of surgically modifying a generic filter to a new case: Woodin's key lemma applies when modifications are done on the range of j , whereas our argument uses a new key lemma to handle modifications done off of the range of j on the ghost coordinates. This work answers a question of Friedman and Honzik [1]. We also discuss an open question involving an analogue of the above result in the case that λ is a singular cardinal.

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1. INTRODUCTION

The behavior of the continuum function $\gamma \mapsto 2^\gamma$ on the regular cardinals was shown, by Easton, to be highly undetermined by the axioms of ZFC. Easton proved [2] that if F is any function from the regular cardinals to the cardinals satisfying $\alpha < \text{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leq F(\beta)$, then there is a cofinality-preserving forcing extension in which $2^\gamma = F(\gamma)$ for every regular cardinal γ . Large cardinal axioms impose additional restrictions on the continuum function on the regular cardinals. For example, if κ is a supercompact cardinal and GCH holds below κ , then GCH holds everywhere. It therefore seems natural to address the question: What functions can be forced to coincide with the continuum function on the regular cardinals while preserving large cardinals? From what hypotheses? In particular, let us consider the following question.

Question 1. Given a λ -supercompact cardinal κ where $\lambda \geq \kappa$ is a cardinal, and assuming GCH, what functions F from the regular cardinals to the cardinals can be forced to equal the continuum function on the interval $[\kappa, \lambda]$ while preserving the λ -supercompactness of κ and preserving cardinals? From what hypotheses?

Menas [3] proved that, assuming GCH, one can force the continuum function to agree at every regular cardinal with any *locally definable*¹ function F satisfying the requirements of Easton's theorem, while preserving all cofinalities and preserving the supercompactness of a cardinal κ . If κ is a Laver-indestructible supercompact cardinal in a model V , as in [4], it easily follows that one can force over this model to achieve any reasonable behavior of the continuum function at and above κ , while preserving the supercompactness of κ . In particular, starting with a Laver-indestructible supercompact cardinal, one can obtain a model with a measurable cardinal at which GCH fails. However, one can also obtain a model with a measurable cardinal at which GCH fails from a much weaker large cardinal assumption. Woodin proved that the existence of a measurable cardinal at which GCH fails is equiconsistent with the existence of an elementary embedding $j : V \rightarrow M$ with critical point κ such that $M^\kappa \subseteq M$ and $j(\kappa) > \kappa^{++}$ (see [5, Theorem 25.1]). Woodin's argument illustrates that under certain conditions, one may perform a type of surgical modification on a generic filter g to obtain g^* in order to meet the lifting criterion, $j''G \subseteq g^*$, and such that g^* remains generic. In Woodin's proof, the modifications made to g in order to obtain g^* only occur on the range of j , and his key lemma shows that such changes are relatively mild in the sense that for a given condition $p \in g$, the set over which modifications are made to obtain $p^* \in g^*$ has size at most κ . Hamkins showed [6] that Woodin's method could be applied to obtain an indestructibility theorem for tall cardinals.

The first author extended Woodin's surgery method to the case of partially supercompact cardinals in [7]. It is shown in [7] that the existence of a λ -supercompact cardinal κ such that $2^\lambda \geq \lambda^{++}$ is equiconsistent with the following hypothesis.

(*) There is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \lambda^{++}$.

The method used in [7] is to, after a suitable preparatory iteration, blow up the size of the powerset of κ using Cohen forcing, in order to achieve $2^\kappa = \lambda^{++}$ and then use

¹A function F is *locally definable* if there is a true sentence ψ and a formula $\varphi(x, y)$ such that for all cardinals γ , if $H_\gamma \models \psi$, then F has a closure point at γ and for all $\alpha, \beta < \gamma$, we have $F(\alpha) = \beta \leftrightarrow H_\gamma \models \varphi(\alpha, \beta)$.

Woodin’s method of surgery to lift the elementary embedding. Thus, one obtains a model in which κ is λ -supercompact and GCH fails at λ , *because* $2^\kappa = \lambda^{++}$. Answering a question posed in [7], Friedman and Honzik [1] used a variant of Sacks forcing for uncountable cardinals to show, from the hypothesis (*), one can obtain a forcing extension in which κ is λ -supercompact, GCH holds on $[\kappa, \lambda)$, and $2^\lambda \geq \lambda^{++}$. The methods of both [7] and [1] leave open the following question, which appears in [1]. Assuming GCH and (*), where $\kappa < \gamma < \lambda$ are regular cardinals, is there a cofinality-preserving forcing extension in which κ remains λ -supercompact, GCH holds on the interval $[\kappa, \gamma)$, and $2^\gamma = \lambda^{++}$? In this article we answer this question, and indeed, provide a full answer to Question 1, by proving the following theorem.

Theorem 1. *Suppose GCH holds, $\kappa < \lambda$ are regular cardinals, and $F : [\kappa, \lambda] \cap \text{REG} \rightarrow \text{CARD}$ is a function such that $\forall \alpha, \beta \in \text{dom}(F)$ $\alpha < \text{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leq F(\beta)$. If there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > F(\lambda)$, then there is a cofinality-preserving forcing extension in which κ remains λ -supercompact and $2^\gamma = F(\gamma)$ for every regular cardinal $\gamma \in [\kappa, \lambda]$.*

The stage κ forcing \mathbb{Q} in our proof will be an Easton-support product forcing for adding $F(\gamma)$ -subsets to γ for each regular cardinal $\gamma \in [\kappa, \lambda]$. Since conditions p in this forcing will have size greater than the critical point of the embedding j under consideration, it follows that $j(p) \neq j''p$ and hence $j(p)$ will have ghost-coordinates. Part of the modification we will perform on a certain generic filter g to obtain g^* with $j''H \subseteq g^*$ will occur off of the range of j on these ghost coordinates. Hence, Woodin’s key lemma does not apply. We will formulate and prove a new key lemma (Lemma 4 below) that allows much of the rest of the argument to be carried out as before. Essentially, our new key lemma shows that the set over which modifications are made to a condition $p \in g$ to obtain $p^* \in g^*$ can be broken up into λ pieces, each of which is in the relevant model.

The hypothesis in Theorem 1 is of optimal consistency strength for the simple reason that if $j : V \rightarrow M$ witnesses that κ is λ -supercompact, then

$$2^\lambda \leq (2^\lambda)^M < j(\kappa).$$

In Section 4 we discuss an easy corollary of Theorem 1 and an open question both addressing the case when λ is a singular cardinal.

2. PRELIMINARIES

2.1. Background material. We assume familiarity with the large cardinal notions of measurable and supercompact cardinals as well as with the characterization of these notions in terms of the existence of nontrivial elementary embeddings from the universe V into transitive inner models $M \subseteq V$. Some familiarity with the techniques of lifting large cardinal embeddings to forcing extensions (see [5, Sections 8 and 9] and see [8, Chapter 21]) is also assumed.

2.2. Supercompactness with tallness. We say that a cardinal κ is θ -tall if $\theta > \kappa$ is an ordinal and there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > \theta$ and $M^\kappa \subseteq M$. We say that κ is λ -supercompact with tallness θ if $\kappa \leq \lambda$ are cardinals, $\theta > \lambda$ is an ordinal, and there is a $j : V \rightarrow M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \theta$. Such cardinals have been used by Gitik, Woodin, Hamkins [6], Cody [7], Friedman-Honzik [1], and others.

The next lemma appears in [7], and is easy to verify by factoring an elementary embedding through the ultrapower by an extender.

Lemma 1. *If κ is λ -supercompact with tallness θ then there is an embedding $j : V \rightarrow M$ witnessing this such that*

$$M = \{j(h)(j''\lambda, \alpha) \mid h : P_\kappa\lambda \times \kappa \rightarrow V \ \& \ \alpha < \theta^\lambda \ \& \ h \in V\}.$$

2.3. Easton's product forcing. Let us quickly review Easton's product forcing and fix some notation. Suppose κ and λ are regular cardinals and $F : [\kappa, \lambda] \cap \text{REG} \rightarrow \text{CARD}$ is a function satisfying Easton's requirements; that is, for all regular cardinals $\alpha, \beta \in [\kappa, \lambda]$ one has

$$\alpha < \text{cf}(F(\alpha)) \text{ and} \tag{E1}$$

$$\alpha < \beta \implies F(\alpha) \leq F(\beta). \tag{E2}$$

Let $\mathbb{Q}_{[\kappa, \lambda]}^F$ denote the Easton support product of Cohen forcing to achieve $2^\gamma = F(\gamma)$ for every regular $\gamma \in [\kappa, \lambda]$, by forcing over a model of GCH. We can regard conditions $p \in \mathbb{Q}_{[\kappa, \lambda]}^F$ as functions satisfying the following.

- Every element in $\text{dom}(p)$ is of the form (γ, α, β) where $\gamma \in [\kappa, \lambda]$ is a regular cardinal, $\alpha < \gamma$, and $\beta < F(\gamma)$.
- (Easton support) For each regular cardinal $\gamma \in [\kappa, \lambda]$ we have

$$|\{(\delta, \alpha, \beta) \in \text{dom}(p) \mid \delta \leq \gamma\}| < \gamma.$$

- $\text{ran}(p) \subseteq \{0, 1\}$.

Let $\text{dom}(\mathbb{Q}_{[\kappa, \lambda]}^F) := \bigcup \{\text{dom}(p) \mid p \in \mathbb{Q}_{[\kappa, \lambda]}^F\}$ and notice that

$$\text{dom}(\mathbb{Q}_{[\kappa, \lambda]}^F) = \{(\delta, \alpha, \beta) \mid \delta \in [\kappa, \lambda] \cap \text{REG}, \alpha < \delta, \text{ and } \beta < F(\delta)\}$$

By [2] we obtain the following.

Lemma 2. *Assuming GCH, forcing with the poset $\mathbb{Q}_{[\kappa, \lambda]}^F$ preserves all cofinalities and achieves $2^\gamma = F(\gamma)$ for every regular cardinal $\gamma \in [\kappa, \lambda]$ while preserving GCH otherwise.*

One may also see [8, Theorem 15.18] for further details regarding the partial order $\mathbb{Q}_{[\kappa, \lambda]}^F$ and Lemma 2.

3. PROOF OF THE MAIN THEOREM

The next lemma will allow us to define the forcing iteration we will use to prove Theorem 1. Notice that if GCH holds and κ, λ, F , and j are as in the statement of Theorem 1 then $F(\lambda)^\lambda = F(\lambda)$ since $\lambda < \text{cf}(F(\lambda))$. Thus, under the hypothesis of Theorem 1, it follows from Lemma 1 that we may assume without loss of generality that $j : V \rightarrow M$ is such that

$$M = \{j(h)(j''\lambda, \alpha) \mid h : P_\kappa\lambda \times \kappa \rightarrow V \ \& \ \alpha < F(\lambda) \ \& \ h \in V\} \tag{3.1}$$

Lemma 3. *Assume GCH and that κ, λ, F , and $j : V \rightarrow M$ are as in the hypothesis of Theorem 1. Furthermore, assume that M is as in (3.1). Then there is a cofinality-preserving forcing extension $V[G]$ such that the following hold.*

- (1) GCH holds in $V[G]$.
- (2) There is a partial function f from κ to V_κ in $V[G]$ such that j lifts to $j : V[G] \rightarrow M[j(G)]$ and $j(f)(\kappa) = F$.

$$(3) \ M[j(G)] = \{j(h)(j''\lambda, \alpha) \mid h : (P_\kappa\lambda)^V \times \kappa \rightarrow V \ \& \ \alpha < F(\lambda) \ \& \ h \in V[G]\}$$

Proof. Fix $\vec{x} = \langle x_\xi \mid \xi < \kappa \rangle$, a well-ordering of V_κ . Then $j(\vec{x}) = \langle z_\xi \mid \xi < j(\kappa) \rangle$ is a well ordering of $V_{j(\kappa)}^M$. Since $M^\lambda \subseteq M$ it is clear that $F \in M$ and hence $F \in V_{j(\kappa)}^M$. Fix $\alpha < j(\kappa)$ such that $F = z_\alpha$. Let \mathbb{F}_κ be Woodin's poset for adding a partial function from κ to κ , defined as follows. Conditions in \mathbb{F}_κ are partial functions $p \subseteq \kappa \times \kappa$ satisfying the following conditions.

- $\gamma \in \text{dom}(p) \implies \gamma < \kappa$ is an inaccessible cardinal and $p''\gamma \subseteq \gamma$.
- For every inaccessible cardinal $\mu < \kappa$ one has $|p \upharpoonright \mu| < \mu$.

The ordering on \mathbb{F}_κ is defined by $p \leq p'$ if and only if $p \supseteq p'$. For a proof that under GCH the poset \mathbb{F}_κ preserves all cofinalities and does not disturb the continuum function, see [9, Theorem 1.3]. Let G be V -generic for \mathbb{F}_κ and let $\tilde{f} = \bigcup G$. Clearly \tilde{f} is a partial function from κ to κ .

Let us show that in M the poset $j(\mathbb{F}_\kappa)$ factors below a condition p_0 as $j(\mathbb{F}_\kappa)/p_0 \cong \mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$ such that \mathbb{F}_{tail} is $\leq F(\lambda)$ -closed in M . Notice that $\{(\kappa, \alpha)\}$ is a condition in \mathbb{F}_κ and that the forcing $j(\mathbb{F}_\kappa)$ can be factored below $\{(\kappa, \alpha)\}$ as $j(\mathbb{F}_\kappa)/\{(\kappa, \alpha)\} \cong \mathbb{F}_\kappa \times \mathbb{F}_{[\gamma_0, j(\kappa)]}$ where γ_0 denotes the least M -inaccessible cardinal greater than α , and $\mathbb{F}_{[\gamma_0, j(\kappa)]}$ is a $\leq \gamma_0$ -closed poset in M . If $\gamma_0 \geq F(\lambda)$ then we can take $p_0 = \{(\kappa, \alpha)\}$ and $\mathbb{F}_{\text{tail}} = \mathbb{F}_{[\gamma_0, j(\kappa)]}$. Assuming $\gamma_0 < F(\lambda)$, let $p_0 = \{(\kappa, \alpha), (\gamma_0, F(\lambda))\}$ and factor $j(\mathbb{F}_\kappa)$ below the condition $p_0 = \{(\kappa, \alpha), (\gamma_0, F(\lambda))\}$ to obtain

$$j(\mathbb{F}_\kappa)/\{(\kappa, \alpha), (\gamma_0, F(\lambda))\} \cong \mathbb{F}_\kappa \times \mathbb{F}_{[\gamma_1, j(\kappa)]}$$

where γ_1 is the least M -inaccessible cardinal greater than $F(\lambda)$ and $\mathbb{F}_{[\gamma_1, j(\kappa)]}$ is $\leq \gamma_1$ -closed in M . In this case we take $\mathbb{F}_{\text{tail}} = \mathbb{F}_{[\gamma_1, j(\kappa)]}$.

Now we will show that there is an M -generic filter K for $j(\mathbb{F}_\kappa)/p_0 \cong \mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$ in $V[G]$ satisfying the lifting criterion $j''G \subseteq K$, and hence j lifts to $j : V[G] \rightarrow M[j(G)]$ (see [5, Proposition 9.1]) where $j(G)$ is the upward closure of K in $j(\mathbb{F}_\kappa)$. This will suffice for (2) because $p_0 \in j(G)$ ensures that $j(\tilde{f})(\kappa) = \alpha$, and then working in $V[G]$, one can define a function f with $\text{dom}(f) \subseteq \kappa$ by $f(\xi) = x_{\tilde{f}(\xi)}$, and this function satisfies $j(f)(\kappa) = z_{j(\tilde{f})(\kappa)} = z_\alpha = F$.

Let us show that K , as above, can be built in $V[G]$. First we show that there is an M -generic filter G_{tail} for \mathbb{F}_{tail} in V . Let X denote the set of all dense subsets of all tails of the forcing \mathbb{F}_κ . It follows that X has size at most $2^\kappa = \kappa^+$ in V . Since M is as in (3.1), every dense subset D of \mathbb{F}_{tail} in M is represented by a function $h : P_\kappa\lambda \times \kappa \rightarrow X$ as $D = j(h)(j''\lambda, \alpha)$ where $\alpha < F(\lambda)$ and $h \in V$. Since there are at most $(2^\kappa)^{\lambda^{<\kappa}} = (2^\kappa)^\lambda = 2^\lambda = \lambda^+$ such functions we can enumerate them as $\langle h_\xi \mid \xi < \lambda^+ \rangle \in V$. Working in V we will build a decreasing sequence $\langle p_\xi \mid \xi < \lambda^+ \rangle$ with $p_\xi \in \mathbb{F}_{\text{tail}}$ such that p_ξ meets every dense subset of \mathbb{F}_{tail} in M represented by some h_ζ for $\zeta \leq \xi$. Assume p_ζ has been constructed for $\zeta < \xi$. At stage $\xi < \lambda$, since $M^\lambda \subseteq M$ in V , we may let $r_\xi \in \mathbb{F}_{\text{tail}}$ be a lower bound of all previously constructed conditions. Since $\langle j(h_\xi)(j''\lambda, \alpha) \mid \alpha < F(\lambda) \rangle \in M$ and \mathbb{F}_{tail} is $\leq F(\lambda)$ -closed in M , it follows that there is a single condition p_ξ below r_ξ meeting every dense subset of \mathbb{F}_{tail} in the sequence $\langle j(h_\xi)(j''\lambda, \alpha) \mid \alpha < F(\lambda) \rangle$. Since every dense subset of \mathbb{F}_{tail} in M has a name that is represented by some function on our list $\langle h_\xi \mid \xi < \lambda^+ \rangle$, we can use $\langle p_\xi \mid \xi < \lambda^+ \rangle$ to generate $G_{\text{tail}} \in V$ an M -generic filter for \mathbb{F}_{tail} . Now let $K = G \times G_{\text{tail}}$. It is easy to verify that K is M -generic for $j(\mathbb{F}_\kappa)/p_0 \cong \mathbb{F}_\kappa \times F_{\text{tail}}$.

We now show that (3) holds. If $x \in M[j(G)]$ then x has a $j(\mathbb{F}_\kappa)$ -name, $\dot{x} \in M$ such that $x = \dot{x}_{j(G)}$. Thus we may write $\dot{x} = j(h)(j''\lambda, \alpha)$ for some $h : P_\kappa\lambda \times \kappa \rightarrow V$, $\alpha < F(\lambda)$, and $h \in V$. Working in $V[G]$ define a function $\tilde{h} : (P_\kappa\lambda)^V \times \kappa \rightarrow V[G]$ by letting $\tilde{h}(z, \xi) = h(z, \xi)_G$ whenever $h(z, \xi)$ is a \mathbb{P} -name and letting $\tilde{h}(z, \xi) = \emptyset$ otherwise. Then $\tilde{h} \in V[G]$ and $j(\tilde{h})(j''\lambda, \alpha) = j(h)(j''\lambda, \alpha)_{j(G)} = \dot{x}_{j(G)} = x$. \square

Let us restate the main theorem.

Theorem 1. *Suppose GCH holds, $\kappa < \lambda$ are regular cardinals, and $F : [\kappa, \lambda] \cap \text{REG} \rightarrow \text{CARD}$ is a function such that $\forall \alpha, \beta \in \text{dom}(F)$ $\alpha < \text{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leq F(\beta)$. If there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > F(\lambda)$, then there is a cofinality-preserving forcing extension in which κ remains λ -supercompact and $2^\gamma = F(\gamma)$ for every regular cardinal $\gamma \in [\kappa, \lambda]$.*

Proof. To simplify notation later on, let \tilde{V} denote the model we start with and let $\tilde{j} : \tilde{V} \rightarrow \tilde{M}$ be an elementary embedding, which is a definable class of \tilde{V} such that (1) $\tilde{M}^\lambda \subseteq \tilde{M}$, (2) F is as in the statement of the theorem, and (3) $\tilde{j}(\kappa) > F(\lambda)$. Since GCH implies $\lambda^{<\kappa} = \lambda$ and $F(\lambda)^\lambda = F(\lambda)$, we can further assume without loss of generality that

$$\tilde{M} = \{\tilde{j}(h)(\tilde{j}''\lambda, \alpha) \mid h : P_\kappa\lambda \times \kappa \rightarrow \tilde{V}, \alpha < F(\lambda), \text{ and } h \in \tilde{V}\}.$$

Now let $V = \tilde{V}[\tilde{G}]$ be the forcing extension of Lemma 3. So \tilde{j} lifts to $j : V \rightarrow M = \tilde{M}[\tilde{j}(\tilde{G})]$ where

$$M = \{j(h)(j''\lambda, \alpha) \mid h : (P_\kappa\lambda)^{\tilde{V}} \times \kappa, \alpha < F(\lambda), \text{ and } h \in V\},$$

and there is a partial function f from κ to V_κ in V such that $j(f)(\kappa) = F$. Working in V , define $Y_f \subseteq \text{dom}(f)$ to be the set of all $\gamma < \kappa$ such that the following properties hold.

- (1) $\gamma \in \text{dom}(f)$ is an inaccessible cardinal and $f''\gamma \subseteq V_\gamma$.
- (2) For some regular cardinal $\lambda_\gamma \in [\gamma, \kappa)$, the value of $f(\gamma)$ is a function from $[\gamma, \lambda_\gamma] \cap \text{REG}$ to CARD satisfying the requirements of Easton's theorem.

Clearly, Y_f must have measure one with respect to the normal measure $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$.

Let $\mathbb{P}_{\kappa+1} = \langle (\mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma) \mid \gamma < \kappa + 1 \rangle$ be the length $\kappa + 1$ Easton support iteration defined as follows.

- If $\gamma \in Y_f$ then let $\dot{\mathbb{Q}}_\gamma = \dot{\mathbb{Q}}_{[\gamma, \lambda_\gamma]}^{f(\gamma)}$ be a \mathbb{P}_γ -name for the Easton-support product $\mathbb{Q}_{[\gamma, \lambda_\gamma]}^{f(\gamma)}$ as defined in $V^{\mathbb{P}_\gamma}$ (see Section 2 above for a definition of the poset $\mathbb{Q}_{[\gamma, \lambda_\gamma]}^{f(\gamma)}$).
- If $\gamma < \kappa$ and $\gamma \notin Y_f$, then $\dot{\mathbb{Q}}_\gamma$ is a \mathbb{P}_γ -name for trivial forcing.
- For the stage κ forcing, let $\dot{\mathbb{Q}}_\kappa = \dot{\mathbb{Q}}_{[\kappa, \lambda]}^F$ be a \mathbb{P}_κ -name for the Easton-support product $\mathbb{Q}_{[\kappa, \lambda]}^F$ as defined in $V^{\mathbb{P}_\kappa}$.

Let $\mathbb{P} := \mathbb{P}_\kappa$ and $\dot{\mathbb{Q}} := \dot{\mathbb{Q}}_\kappa$. Let $G * H$ be V -generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Standard arguments show that $\mathbb{P} * \dot{\mathbb{Q}}$ preserves cofinalities and that in $V[G][H]$ one has $2^\delta = F(\delta)$ for every regular cardinal δ in $[\kappa, \lambda]$. Our argument follows Woodin's: we show that there is a further cofinality-preserving forcing extension $V[G][H][g_0]$ in which κ

remains λ -supercompact, and the desired continuum function is preserved on the interval $[\kappa, \lambda]$.

3.1. Lifting the embedding through \mathbb{P} . By elementarity, the forcing $j(\mathbb{P})$ can be factored in M as $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}}_{[\kappa, \lambda, \kappa]}^{j(f)(\kappa)} * \dot{\mathbb{P}}_{\text{tail}}$. Since $j(f)(\kappa) = F$ by Lemma 3(3), and since $M^\lambda \subseteq M$, it follows that

$$j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}}_{[\kappa, \lambda]}^F * \dot{\mathbb{P}}_{\text{tail}}.$$

Since the next stage of nontrivial forcing in $j(\mathbb{P})$ beyond κ must be beyond $F(\lambda)$, it follows that $\mathbb{P}_{\text{tail}} := (\dot{\mathbb{P}}_{\text{tail}})_{G * H}$ is $\leq F(\lambda)$ -closed in $M[G][H]$.

Now we will show that there is an $M[G][H]$ -generic filter for \mathbb{P}_{tail} in $V[G][H]$. The argument is similar to that in the proof of Lemma 3. For $\xi < \kappa$ let X_ξ denote the collection of all nice \mathbb{P}_ξ -names for dense subsets of the tail $\dot{\mathbb{P}}_{\xi, \kappa}$ of the iteration \mathbb{P}_κ . It follows that X_ξ has size at most 2^κ and hence $X := \bigcup \{X_\xi \mid \xi < \kappa\}$ has size at most 2^κ . Suppose $D \in M[G][H]$ is a dense subset of \mathbb{P}_{tail} and let $\dot{D} \in M$ be a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{[\kappa, \lambda]}^F$ -name for D . Without loss of generality we may assume that $\dot{D} \in j(X)$ and hence there is a function $h_{\dot{D}} : (P_\kappa \lambda)^{\tilde{V}} \times \kappa \rightarrow X$ in V with $\dot{D} = j(h)(j''\lambda, \alpha)$ for some $\alpha < F(\lambda)$. In other words, every dense subset of \mathbb{P}_{tail} in $M[G][H]$ has a $\mathbb{P} * \dot{\mathbb{Q}}$ -name that is represented by a function $(P_\kappa \lambda)^{\tilde{V}} \times \kappa \rightarrow X$ in V . Since there are at most $(2^\kappa)^{\lambda^{<\kappa}} = 2^\lambda = \lambda^+$ -such functions we can enumerate them as $\langle h_\xi \mid \xi < \lambda^+ \rangle \in V$. Working in $V[G][H]$ we build a decreasing sequence $\langle p_\xi \mid \xi < \lambda^+ \rangle$ with $p_\xi \in \mathbb{P}_{\text{tail}}$ meeting every dense subset of \mathbb{P}_{tail} in $M[G][H]$ as follows. Assume p_ζ has been constructed for $\zeta < \xi$. At stage $\xi < \lambda$, since $M[G][H]^\lambda \subseteq M[G][H]$ in $V[G][H]$, we may let $r_\xi \in \mathbb{P}_{\text{tail}}$ be a lower bound of all previously constructed conditions. Since $\langle j(h_\xi)(j''\lambda, \alpha)_{G * H} \mid \alpha < F(\lambda) \rangle \in M[G][H]$ and \mathbb{P}_{tail} is $\leq F(\lambda)$ -closed in $M[G][H]$, it follows that there is a single condition p_ξ below r_ξ meeting every dense subset of \mathbb{P}_{tail} in the sequence $\langle j(h_\xi)(j''\lambda, \alpha)_{G * H} \mid \alpha < F(\lambda) \rangle$. Since every dense subset of \mathbb{P}_{tail} in $M[G][H]$ has a name that represented by some function on our list $\langle h_\xi \mid \xi < \lambda^+ \rangle$, we can use $\langle p_\xi \mid \xi < \lambda^+ \rangle$ to generate an $M[G][H]$ -generic filter, call it $G_{\text{tail}} \in V[G][H]$.

Since conditions in \mathbb{P}_κ have bounded support, it follows that $j''G \subseteq G * H * G_{\text{tail}}$, and thus we may lift the embedding to $j : V[G] \rightarrow M[j(G)]$ in $V[G][H]$ where $j(G) = G * H * G_{\text{tail}}$ (as before, see [5, Proposition 9.1]). We have

$$M[j(G)] = \{j(h)(j''\lambda, \alpha) \mid h : (P_\kappa \lambda)^{\tilde{V}} \times \kappa \rightarrow V[G], \alpha < F(\lambda), \text{ and } h \in V[G]\}.$$

3.2. Obtaining an $M[j(G)]$ -generic filter for $j(\mathbb{Q}) = j(\mathbb{Q}_{[\kappa, \lambda]}^F)$. We will factor the embedding $j : V[G] \rightarrow M[j(G)]$ through an ultrapower embedding j_0 , force over $V[G][H]$ with $j_0(\mathbb{Q})$ to obtain g_0 , and then transfer g_0 to an $M[j(G)]$ -generic filter g for $j(\mathbb{Q})$. We will also show that forcing with $j_0(\mathbb{Q})$ over $V[G][H]$ preserves cardinals and does not disturb the continuum function below, or at, λ .

Let $X = \{j(h)(j''\lambda, F(\lambda)) \mid h : (P_\kappa \lambda)^{\tilde{V}} \times \kappa \rightarrow V[G] \text{ where } h \in V[G]\}$. Then it follows that $X \prec M[j(G)]$. Let $k : M'_0 \rightarrow M[j(G)]$ be the inverse of the Mostowski collapse $\pi : X \rightarrow M'_0$ and let $j_0 : V[G] \rightarrow M'_0$ be defined by $j_0 := k^{-1} \circ j$. It follows that j_0 is the ultrapower embedding by the measure $U_0 := \{X \subseteq (P_\kappa \lambda)^{\tilde{V}} \times \kappa \mid (j''\lambda, F(\lambda)) \in j(X)\}$ where $U_0 \in V[G][H]$. By elementarity, M'_0 is of the form $M_0[j_0(G)]$, where $M_0 \subseteq M'_0$ and $j_0(G) \subseteq j_0(\mathbb{P}) \in M'_0$ is M_0 -generic. Furthermore,

$j_0(G) = G * H_0 * G_{\text{tail}}^0$ where H_0 is $M_0[G]$ -generic for $\pi(\mathbb{Q}_{[\kappa, \lambda]}^F)$ and G_{tail}^0 is $M[G][H_0]$ -generic for the tail of the iteration $j_0(\mathbb{P})$ above stage κ . The following diagram is commutative.

$$\begin{array}{ccc} V[G] & \xrightarrow{j} & M[j(G)] \\ & \searrow j_0 & \uparrow k \\ & & M_0[j_0(G)] \end{array}$$

It follows that j_0 is a class of $V[G][H_0]$, $M_0[j_0(G)]$ is closed under λ -sequences in $V[G][H_0]$, and that $j_0(\kappa) > \pi(F(\lambda))$. Furthermore, since $j''\lambda \in X$ it follows that $X^\lambda \subseteq X$ in $V[G][H]$ and hence $\lambda + 1 \subseteq X$. Thus, $\pi(\lambda) = \lambda$ and this implies that $\pi(\mathbb{Q}_{[\kappa, \lambda]}^F) = \mathbb{Q}_{[\kappa, \lambda]}^{\pi(F)}$. Let g_0 be $V[G][H]$ -generic for $j_0(\mathbb{Q})$.

The next claim shows that forcing with $j_0(\mathbb{Q})$ over $V[G][H]$ preserves cardinals and does not disturb the continuum function below, or at, λ .

Claim 1. $j_0(\mathbb{Q})$ is $\leq \lambda$ -distributive and λ^{++} -c.c. in $V[G][H]$.

Proof of Claim 1. First we will demonstrate the distributivity. By elementarity, $j_0(\mathbb{Q})$ is $\leq \lambda$ -closed in $M_0[j_0(G)]$. Since $G_{\text{tail}}^0 \in V[G][H_0]$, it follows that $j_0(\mathbb{Q}) \in V[G][H_0]$ and since $M_0[j_0(G)]$ is closed under λ -sequences in $V[G][H_0]$, it follows that $j_0(\mathbb{Q})$ is $\leq \lambda$ -closed in $V[G][H_0]$. Notice that H_0 is $V[G]$ -generic for $\pi(\mathbb{Q}_{[\kappa, \lambda]}^F) = \mathbb{Q}_{[\kappa, \lambda]}^{\pi(F)}$ and that the quotient forcing $\mathbb{Q}_{[\kappa, \lambda]}^F/H_0$ is λ^+ -c.c. in $V[G][H_0]$. Since $\leq \lambda$ -closed forcing remains $\leq \lambda$ -distributive after λ^+ -c.c. forcing, we see that $j_0(\mathbb{Q})$ is $\leq \lambda$ -distributive in $V[G][H]$.

Now let us show that $j_0(\mathbb{Q})$ is λ^{++} -c.c. in $V[G][H]$. Each $p \in j_0(\mathbb{Q})$ can be written as $p = j(h_p)(j''\lambda, F(\lambda))$ for some function $h_p : (P_\kappa \lambda)^{\bar{V}} \times \kappa \rightarrow \mathbb{Q}$ with $h_p \in V[G]$. For each $p \in j_0(\mathbb{Q})$ the domain of h_p has size λ in $V[G]$, and thus h_p leads to $\bar{h}_p : \lambda \rightarrow \mathbb{Q}$, which can be viewed as a condition in the full support product of λ copies of \mathbb{Q} taken in $V[G]$. Let us denote this product by $\bar{\mathbb{Q}}$. We will show that $j_0(\mathbb{Q})$ is λ^{++} -c.c. in $V[G][H]$ by arguing that $\bar{\mathbb{Q}}$ is λ^{++} -c.c. in $V[G][H]$ and that an antichain of $j_0(\mathbb{Q})$ of size λ^{++} in $V[G][H]$ would lead to an antichain of $\bar{\mathbb{Q}}$ of size λ^{++} in $V[G][H]$.

We now briefly describe how to prove that $\bar{\mathbb{Q}}$ is λ^{++} -c.c. in $V[G][H]$. An easy delta-system argument shows that $\bar{\mathbb{Q}}$ is λ^{++} -c.c. in $V[G]$. Suppose A is an antichain of size δ of $\bar{\mathbb{Q}}$ in $V[G][H]$. We will show that A leads to an antichain of $\mathbb{Q} \cong \mathbb{Q} \times \mathbb{Q}$ in $V[G]$ of size δ . Let

$$q \Vdash \dot{A} \text{ is an antichain of } \bar{\mathbb{Q}} \text{ and } \dot{f} : \delta \rightarrow \dot{A} \text{ is bijective}$$

where $q \in \mathbb{Q}$ and $\dot{A}_H = A$. For each $\alpha < \delta$ let $q_\alpha \leq q$ be such that $q_\alpha \Vdash \dot{f}(\alpha) = \dot{p}_\alpha$ where $p_\alpha \in \bar{\mathbb{Q}}$. We have $\bar{\mathbb{Q}} \cong \mathbb{Q} \times \mathbb{Q}$ in $V[G]$, and it is easy to check that $W := \{(q_\alpha, p_\alpha) \mid \alpha < \delta\}$ is an antichain of $\bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$ in $V[G]$ of size δ . This shows that $\bar{\mathbb{Q}}$ is λ^{++} -c.c. in $V[G][H]$.

It remains to show that an antichain of $j_0(\mathbb{Q})$ in $V[G][H]$ with size λ^{++} would lead to an antichain of $\bar{\mathbb{Q}}$ in $V[G][H]$ of size λ^{++} . Suppose A is an antichain of $j_0(\mathbb{Q})$ with size δ in $V[G][H]$. Each $p \in A$ is of the form $j_0(h_p)(j''\lambda, F(\lambda))$ where $h_p : P_\kappa \lambda \times \kappa \rightarrow \mathbb{Q}$. As mentioned above, each h_p leads to a condition $\bar{h}_p \in \bar{\mathbb{Q}}$. It is easy to check that $\bar{A} := \{\bar{h}_p \mid p \in A\}$ is an antichain of $\bar{\mathbb{Q}}$. \square

Let us now show that g_0 can be transferred along k to an $M[j(G)]$ -generic filter for $j(\mathbb{Q})$.

Claim 2. $k''g_0 \subseteq j_0(\mathbb{Q})$ generates an $M[j(G)]$ -generic filter for $j(\mathbb{Q})$.

Proof. Suppose $D \in M[j(G)]$ is an open dense subset of $j(\mathbb{Q})$ and let $D = j(h)(j''\lambda, \alpha)$ for some $h \in V[G]$ with $\text{dom}(h) = (P_\kappa\lambda)^{\tilde{V}} \times \kappa$ and $\alpha < F(\lambda)$. Without loss of generality, let us assume that every element of the range of h is a dense subset of \mathbb{Q} in $V[G]$. We have $D = j(h)(j''\lambda, \alpha) = k(j_0(h))(j''\lambda, \alpha)$. Define a function $\tilde{h} \in M_0[j_0(G)]$ with $\text{dom}(\tilde{h}) = \pi(F(\lambda))$ by $\tilde{h}(\xi) = j_0(h)(j_0''\lambda, \xi)$. Then $\text{dom}(k(\tilde{h})) = k(\pi(F(\lambda))) = F(\lambda)$ and we have $D = k(\tilde{h})(\alpha)$. Now the range of \tilde{h} is a collection of $\pi(F(\lambda))$ open dense subsets of $j_0(\mathbb{Q})$. Since $j_0(\mathbb{Q})$ is $\leq \pi(F(\lambda))$ -distributive in $M_0[j_0(G)]$, one sees that $\tilde{D} = \bigcap \text{ran}(\tilde{h})$ is an open dense subset of $j_0(\mathbb{Q})$. Hence there is a condition $p \in g_0 \cap \tilde{D}$ and by elementarity, $k(p) \in k''g_0 \cap k(\tilde{D}) \subseteq D$. \square

3.3. Performing surgery. With the $M[j(G)]$ -generic filter g for $j(\mathbb{Q})$ in hand, we surgically modify g to obtain g^* with $j''H \subseteq g^*$, and then argue that g^* remains an $M[j(G)]$ -generic filter for $j(\mathbb{Q})$.

Now let us define g^* . For $p \in g \subseteq j(\mathbb{Q}) = j(\mathbb{Q}_{[\kappa, \lambda]}^F)$, define a new partial function p^* with $\text{dom}(p^*) = \text{dom}(p)$ by letting p^* be equal to p unless p contradicts $j''H$, in which case we flip the appropriate bits so that p^* is compatible with the elements of $j''H$. More precisely, working in $V[G][H]$, let Q be the partial function with $\text{dom}(Q) \subseteq \text{dom}(j(\mathbb{Q}_{[\kappa, \lambda]}^F))$ defined by $Q = \bigcup j''H$. Given $p \in g$, let p^* be the condition in $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$ obtained from p by altering p on $\text{dom}(p) \cap \text{dom}(Q)$ so that p^* agrees with Q . Let

$$g^* = \{p^* \mid p \in g\}.$$

Clearly, $j''H \subseteq g^*$, and it remains to argue that g^* remains an $M[j(G)]$ -generic filter for $j(\mathbb{Q})$.

Since conditions $p \in \mathbb{Q}_{[\kappa, \lambda]}^F$ can have size greater than the critical point of j , it follows that $j(p)$ need not equal $j''p$. Thus, some of the modifications we made in obtaining g^* occurred off of the range of j on the ghost coordinates, and so Woodin's key lemma does not apply. We will use the following lemma to show that g^* remains an $M[j(G)]$ -generic filter.

Lemma 4 (New Key Lemma). *Suppose $B \in M[j(G)]$ with $B \subseteq j(\text{dom}(\mathbb{Q}_{[\kappa, \lambda]}^F))$ and $|B|^{M[j(G)]} \leq j(\lambda)$. Then the set*

$$\mathcal{I}_B = \{\text{dom}(j(q)) \cap B \mid q \in \mathbb{Q}_{[\kappa, \lambda]}^F\}$$

has size at most λ in $V[G][H]$.

Proof. Let B be as in the statement of the lemma and let $B = j(h)(j''\lambda, \alpha)$ where $h : (P_\kappa\lambda)^{\tilde{V}} \times \kappa \rightarrow P_{\lambda^+}(\text{dom}(\mathbb{Q}_{[\kappa, \lambda]}^F))^{V[G]}$, $\alpha < F(\lambda)$, and $h \in V[G]$. Then $\bigcup \text{ran}(h)$ is a subset of $\text{dom}(\mathbb{Q}_{[\kappa, \lambda]}^F)$ in $V[G]$ with $|\bigcup \text{ran}(h)|^{V[G]} \leq \lambda$. Since $V[G] \models \lambda^{<\lambda} = \lambda$ by our GCH assumption, it will suffice to show that

$$\mathcal{I}_B \subseteq \{j(d) \cap B \mid d \in P_\lambda(\bigcup \text{ran}(h))^{V[G]}\}.$$

Suppose $\text{dom}(j(q)) \cap B \in \mathcal{I}_B$ where $q \in \mathbb{Q}_{[\kappa, \lambda]}^F$. We will show that $\text{dom}(j(q)) \cap B = j(d) \cap B$ for some $d \in P_\lambda(\bigcup \text{ran}(h))^{V[G]}$. Let $d := \text{dom}(q) \cap \bigcup \text{ran}(h)$, then

$\text{dom}(j(q)) \cap B = j(d) \cap B$ since

$$j(d) = \text{dom}(j(q)) \cap \bigcup \text{ran}(j(h)) \supseteq \text{dom}(j(q)) \cap B.$$

□

First let us show that the elements of g^* are conditions in $j(\mathbb{Q}) = j(\mathbb{Q}_{[\kappa, \lambda]}^F)$. Suppose $p \in g^*$ and let $\mathcal{I}_{\text{dom}(p)}$ be the collection of all possible intersections of $\text{dom}(p)$ with the domains of conditions $j(q)$ where $q \in H$. In other words,

$$\mathcal{I}_{\text{dom}(p)} := \{\text{dom}(j(q)) \cap \text{dom}(p) \mid q \in \mathbb{Q}_{[\kappa, \lambda]}^F\}.$$

Since p is a condition in $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$ we have $|\text{dom}(p)|^{M[j(G)]} < j(\lambda)$ and hence by Lemma 4, it follows that $V[G][H] \models |\mathcal{I}_{\text{dom}(p)}| \leq \lambda$. Let $\langle I_\alpha \mid \alpha < \lambda \rangle \in V[G][H]$ be an enumeration of $\mathcal{I}_{\text{dom}(p)}$. For each $\alpha < \lambda$ choose a particular $q_\alpha \in H$ with $\text{dom}(j(q_\alpha)) \cap \text{dom}(p) = I_\alpha$. Since $M[j(G)]$ is closed under λ -sequences in $V[G][H]$, it follows that $\{j(q_\alpha) \mid \alpha < \lambda\}$ is a directed subset of $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$ in $M[j(G)]$. Notice that $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$ is $< j(\kappa)$ -directed closed in $M[j(G)]$, and so we may let $\bar{p} := \bigcup \{j(q_\alpha) \mid \alpha < \lambda\}$ be the corresponding partial master condition in $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$. Since $p, \bar{p} \in M[j(G)]$ one can easily see that $p^* \in M[j(G)]$ since it can be obtained by modifying p to agree with \bar{p} .

Now we will show that g^* remains $M[j(G)]$ -generic for $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$. Suppose $A \in M[j(G)]$ is a maximal antichain of $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$. Since $\mathbb{Q}_{[\kappa, \lambda]}^F$ is λ^+ -c.c. in $V[G]$, it follows by elementarity that the set $B := \bigcup \{\text{dom}(r) \mid r \in A\}$ has size at most $j(\lambda)$ in $M[j(G)]$. Hence it follows from Lemma 4 that the set of intersections

$$\mathcal{I}_B := \{\text{dom}(j(q)) \cap B \mid q \in \mathbb{Q}_{[\kappa, \lambda]}^F\}$$

satisfies $V[G][H] \models |\mathcal{I}_B| \leq \lambda$. Let $\langle E_\alpha \mid \alpha < \lambda \rangle \in V[G][H]$ be an enumeration of \mathcal{I}_B . For each $\alpha < \lambda$, choose $q_\alpha \in H$ with $\text{dom}(j(q_\alpha)) \cap B = E_\alpha$. Choose $r_\alpha \in g$ with $\text{dom}(r_\alpha) = \text{dom}(j(q_\alpha))$. Define

$$\Delta_\alpha := \{(\delta, \xi, \zeta) \in \text{dom}(r_\alpha) \mid r_\alpha(\delta, \xi, \zeta) \neq j(q_\alpha)(\delta, \xi, \zeta)\}.$$

Clearly, $\Delta_\alpha \in M[j(G)]$ since $r_\alpha, j(q_\alpha) \in M[j(G)]$. Then $\langle \Delta_\alpha \mid \alpha < \lambda \rangle \in M[j(G)]$ because $M[j(G)]$ is closed under λ -sequences in $V[G][H]$, and hence

$$\Delta := \bigcup \{\Delta_\alpha \mid \alpha < \lambda\}$$

is in $M[j(G)]$. The automorphism $\pi_\Delta : j(\mathbb{Q}_{[\kappa, \lambda]}^F) \rightarrow j(\mathbb{Q}_{[\kappa, \lambda]}^F)$ that flips bits of conditions over Δ is in $M[j(G)]$ since $\Delta \in M[j(G)]$. This implies that $\pi_\Delta^{-1}(A)$ is a maximal antichain of $j(\mathbb{Q}_{[\kappa, \lambda]}^F)$ in $M[j(G)]$, and thus, by the $M[j(G)]$ -genericity of g , it follows that there is a condition $r \in g \cap \pi_\Delta^{-1}(A)$. Furthermore, $r^* = \pi_\Delta(r) \in g^*$ and r^* meets A . Hence g^* is $M[j(G)]$ -generic.

This shows that j lifts to $j : V[G][H] \rightarrow M[j(G)][j(H)]$ where $j(G) = G * H * G_{\text{tail}}$ and $j(H) = g^*$ in $V[G][H][g_0]$.

3.4. Lifting j through $j_0(\mathbb{Q}_{[\kappa, \lambda]}^F)$. By Claim 1 above, $j_0(\mathbb{Q}_{[\kappa, \lambda]}^F)$ is $\leq \lambda$ -distributive in $V[G][H]$. By an easy argument involving intersecting λ open dense subsets of $j_0(\mathbb{Q}_{[\kappa, \lambda]}^F)$ (see [5, Proposition 15.1]), one can show that $j''g_0$ generates an $M[j(G)][j(H)]$ -generic filter for $j(j_0(\mathbb{Q}_{[\kappa, \lambda]}^F))$. Thus j lifts to $j : V[G][H][g_0] \rightarrow M[j(G)][j(H)][j(g_0)]$ in $V[G][H][g_0]$ witnessing that κ is λ -supercompact in $V[G][H][g_0]$. □

4. A COROLLARY AND AN OPEN QUESTION

In this section we will discuss the extent to which our methods for proving Theorem 1 can be extended to cases in which κ is λ -supercompact where λ is a singular cardinal. First, we have an easy Corollary to Theorem 1.

Corollary 2. *Suppose GCH holds and there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that for some singular cardinal $\lambda > \kappa$ with $\text{cf}(\lambda) < \kappa$ one has (1) $M^\lambda \subseteq M$, (2) $F : [\kappa, \lambda] \cap \text{REG} \rightarrow \text{CARD}$ is a function satisfying the requirements of Easton's theorem, and (3) $j(\kappa) > \sup\{F(\gamma) \mid \gamma \in [\kappa, \lambda] \cap \text{REG}\}^+$. Then there is a cofinality-preserving forcing extension in which κ remains λ -supercompact and for every regular cardinal $\gamma \in [\kappa, \lambda]$ one has $2^\gamma = F(\gamma)$.*

One can see that Corollary 2 follows directly from Theorem 1 by extending the function F to \tilde{F} with $\text{dom}(\tilde{F}) = [\kappa, \lambda^+] \cap \text{REG}$ as follows. If F is eventually constant on $[\kappa, \lambda] \cap \text{REG}$ then let $\tilde{F}(\lambda^+)$ be this constant value and $\tilde{F} \upharpoonright [\kappa, \lambda] \cap \text{REG} = F$. It follows that the embedding j in the hypothesis of Corollary 2 witnesses that κ is $\lambda^{<\kappa} = \lambda^+$ -supercompact, and thus one can apply Theorem 1 to j and \tilde{F} in order to obtain a forcing extension in which κ is λ -supercompact and $2^\gamma = F(\gamma)$ for all $\gamma \in [\kappa, \lambda] \cap \text{REG}$. In fact, in this case, the hypothesis in Corollary 2(3) can be weakened to $j(\kappa) > \sup\{F(\gamma) \mid \gamma \in [\kappa, \lambda] \cap \text{REG}\}$. For the remaining case in which F is not eventually constant, we may extend F to \tilde{F} with $\text{dom}(\tilde{F}) = [\kappa, \lambda^+] \cap \text{REG}$ by letting $\tilde{F}(\lambda^+) = \sup\{F(\gamma) \mid \gamma \in [\kappa, \lambda] \cap \text{REG}\}^+$ and then apply Theorem 1.

This suggests the following natural question, which our methods do not seem to answer.

Question 2. Suppose GCH holds and κ is λ -supercompact where λ is a singular cardinal with $\text{cf}(\lambda) \geq \kappa$ and $F : [\kappa, \lambda] \cap \text{REG} \rightarrow \text{CARD}$ is a function such that $\forall \alpha, \beta \in \text{dom}(F) \alpha < \text{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leq F(\beta)$. Suppose the λ -supercompactness of κ is witnessed by $j : V \rightarrow M$ and $j(\kappa) > \sup\{F(\gamma) \mid \gamma \in [\kappa, \lambda] \cap \text{REG}\}^+$. Is there a cofinality-preserving forcing extension preserving the λ -supercompactness of κ in which $2^\gamma = F(\gamma)$ for all regular cardinals $\gamma \in [\kappa, \lambda] \cap \text{REG}$?

Under the assumption of Question 2 we have $\lambda^{<\kappa} = \lambda$. Hence, it seems unlikely that there is an easy argument, similar to the above argument for Corollary 2, that would answer Question 2. A more promising strategy for answering Question 2 is to improve Lemma 4 to allow for the surgery argument to be carried out. The main obstacles to carrying out this strategy seem to be that if λ is singular then conditions in the Easton-support forcing $\mathbb{Q}_{[\kappa, \lambda]}^F$ can have support cofinal in λ and $\mathbb{Q}_{[\kappa, \lambda]}^F$ is merely λ^{++} -c.c.

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