Definitions:
A relation \( R \) on a set \( A \) is called **reflexive** if \((a, a) \in R\) for every element \( a \in A \).

A relation \( R \) on a set \( A \) is called **symmetric** if \((b, a) \in R\) whenever \((a, b) \in R\) for all \( a, b \in A \).

A relation \( R \) on a set \( A \) is called **transitive** if whenever \((a, b) \in R\) and \((b, c) \in R\), then \((a, c) \in R\) for \( a, b, c \in A \).

Representing Relations

We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs (digraphs)**.

- If \( R \) is a relation from \( A = \{a_1, a_2, \ldots, a_m\} \) to \( B = \{b_1, b_2, \ldots, b_n\} \), then \( R \) can be represented by the zero-one matrix \( M_R = [m_{ij}] \) with
  - \( m_{ij} = 1 \), if \((a_i, b_j) \in R\), and
  - \( m_{ij} = 0 \), if \((a_i, b_j) \notin R\).

- Note that for creating this matrix we first need to list the elements in \( A \) and \( B \) in a particular, but arbitrary order.

**Example:** How can we represent the relation \( R \) defined between the set \( A = \{1, 2, 3\} \) and set \( B = \{1, 2\} \) where \( R = \{(2, 1), (3, 1), (3, 2)\} \) as a zero-one matrix?

**Solution:** The matrix \( M_R \) is given by

\[
M_R = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
1 & 1
\end{bmatrix}
\]
Representing Relations

• What do we know about the matrices representing a **relation on a set** (a relation from A to A)?
• They are **square** matrices.

• What do we know about matrices representing **reflexive** relations?
• All the elements on the **diagonal** of such matrices $M_{\text{ref}}$ must be **1s**.

\[
M_{\text{ref}} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}
\]

**Zero-One Reflexive, Symmetric**

• **Terms**: Reflexive, non-reflexive, irreflexive, symmetric, asymmetric, and antisymmetric.
  – These relation characteristics are very easy to recognize by inspection of the zero-one matrix.

\[
\begin{bmatrix}
1 & \text{anything} & \text{anything} & \text{anything} \\
\text{anything} & 0 & \text{anything} & \text{anything} \\
\text{anything} & \text{anything} & 0 & \text{anything} \\
\text{anything} & \text{anything} & \text{anything} & 1
\end{bmatrix}
\]

  **Reflexive**: all 1’s on diagonal  \hspace{1cm}  **Irreflexive**: all 0’s on diagonal  \hspace{1cm}  **Symmetric**: all identical across diagonal  \hspace{1cm}  **Antisymmetric**: all 1’s are across from 0’s

**Representing Relations**

• What do we know about the matrices representing **symmetric relations**?
• These matrices are symmetric, that is, $M_R = (M_R)^t$.

\[
M_R = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1
\end{bmatrix}
\]

• **symmetric matrix**, symmetric relation.

**Representing Relations**

• The Boolean operations **join** and **meet** can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

• To obtain the **join** of two zero-one matrices, we apply the Boolean “or” function to all corresponding elements in the matrices.

• To obtain the **meet** of two zero-one matrices, we apply the Boolean “and” function to all corresponding elements in the matrices.
Representing Relations Using Matrices

- **Example:** Let the relations R and S be represented by the matrices

\[
M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\]

- What are the matrices representing \(R \cup S\) and \(R \cap S\)?
- **Solution:** These matrices are given by

\[
M_{R \cup S} = M_R \cup M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R \cap S} = M_R \land M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

- Do you remember the **Boolean product** of two zero-one matrices?
- Let \(A = [a_{ij}]\) be an \(m \times k\) zero-one matrix and \(B = [b_{ij}]\) be a \(k \times n\) zero-one matrix.
- Then the **Boolean product** of A and B, denoted by \(A \circ B\), is the \(m \times n\) matrix with \((i, j)\)th entry \([c_{ij}]\), where

\[
c_{ij} = (a_{i1} \land b_{1j}) \lor (a_{i2} \land b_{2j}) \lor \ldots \lor (a_{ik} \land b_{kj}).
\]

- \(c_{ij} = 1\) if and only if at least one of the terms \((a_{in} \land b_{nj}) = 1\) for some \(n\); otherwise \(c_{ij} = 0\).

- Let us now assume that the zero-one matrices \(M_A = [a_{ij}], M_B = [b_{ij}],\) and \(M_C = [c_{ij}]\) represent relations A, B, and C, respectively.
- **Remember:** For \(M_C = M_A \circ M_B\) we have:

\(-c_{ij} = 1\) if and only if at least one of the terms \((a_{in} \land b_{nj}) = 1\) for some \(n\); otherwise \(c_{ij} = 0\).

- In terms of the **relations**, this means that C contains a pair \((x_n, z_j)\) if and only if there is an element \(y_n\) such that \((x_n, y_n)\) is in relation A and \((y_n, z_j)\) is in relation B.

- Therefore, \(C = B \circ A\) (**composite** of A and B).
Representing Relations Using Matrices

**Example:** Find the matrix representing $R^2$, where the matrix representing $R$ is given by

$$
M_R = \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
$$

**Solution:** The matrix for $R^2$ is given by

$$
M_{R^2} = M_R \cdot M_R = \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix}
$$

How did we get this one? Well, ...

Representing Relations Using Digraphs

**Definition:** A directed graph, or digraph, consists of a set $V$ of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called edges (or arcs).

- The vertex $a$ is called the initial vertex of the edge $(a, b)$, and the vertex $b$ is called the terminal vertex of this edge.

- We can use arrows to display graphs.

Using Directed Graphs

**Def.** A directed graph or digraph $G = (V_G, E_G)$ is a set $V_G$ of vertices (nodes) with a set $E_G \subseteq V_G \times V_G$ of edges (arcs, links). Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R: A \times B$ can be represented as a graph $G_R = (V_G = A \cup B, E_G = R)$.

Matrix representation $M_R$: Graph representation $G_R$:

<table>
<thead>
<tr>
<th></th>
<th>Susan</th>
<th>Mary</th>
<th>Sally</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joe</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Fred</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Mark</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Digraph Reflexive, Symmetric

• It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.

| Reflexive: Every node has a self-loop | Irreflexive: No node links to itself |
| Symmetric: Every link is bidirectional | Antisymmetric: No link is bidirectional |

These are asymmetric & non-antisymmetric
These are non-reflexive & non-irreflexive

Representing Relations Using Digraphs

• Obviously, we can represent any relation $R$ on a set $A$ by the digraph with $A$ as its vertices and all pairs $(a, b) \in R$ as its edges.

• Vice versa, any digraph with vertices $V$ and edges $E$ can be represented by a relation on $V$ containing all the pairs in $E$.

• This one-to-one correspondence between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa. This then means that digraphs are sets, and that all the set operations apply. We’ll use it in closures which come next!

Closures of Relations, or Relational Closures

• Three types we will study
  
  – Reflexive  
    • Easy
  
  – Symmetric  
    • Easy
  
  – Transitive  
    • Hard

Closures of Relations

• Def. For any property $X$, the “$X$ closure” of a set $A$ is defined as the “smallest” superset of $A$ that has the given property.

The reflexive closure of a relation $R$ on $A$ is obtained by adding $(a, a)$ to $R$ for each $a \in A$. i.e., it is $R \cup I_A$

The symmetric closure of $R$ is obtained by adding $(b, a)$ to $R$ for each $(a, b)$ in $R$. i.e., it is $R \cup R^T$ (note in book is $R^{-1}$ used)

• The transitive closure or connectivity relation of $R$ is obtained by repeatedly adding $(a, c)$ to $R$ for each $(a, b), (b, c)$ in $R$, i.e., it is

$$R^* = \bigcup_{n \in \mathbb{Z}^+} R^n = R \lor R^2 \lor R^3 \cdots \lor R^{n-1} \lor R^n$$
**Reflexive closure**

- Consider a relation \( R \):
  - Note that it is not reflexive

- We want to add edges to make the relation reflexive

- By adding those edges, we have made a non-reflexive relation \( R \) into a reflexive relation

- This new relation is called the reflexive closure of \( R \)

**Reflexive closure example**

- Let \( R \) be a relation on the set \( \{ 0, 1, 2, 3 \} \) containing the ordered pairs \((0,1), (1,1), (1,2), (2,0), (2,2), \) and \((3,0)\)
- What is the reflexive closure of \( R \)?
- We add all pairs of edges \((a,a)\) that do not already exist

**Reflexive closure example with matrices**

- Let \( R \) be a relation on the set \( \{ 0, 1, 2, 3 \} \) containing the ordered pairs \((0,1), (1,1), (1,2), (2,0), (2,2), \) and \((3,0)\)
- What is the reflexive closure of \( R \)?
- We ‘add’ a diagonal matrix with ones, called also identity matrix

\[
\begin{align*}
R &= \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]

\[
\Delta = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
R \cup \Delta = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

We add entries: \((0,0), (3,3)\)
Symmetric closure

- Consider a relation $R$:
  - Note that it is not symmetric
- We want to add edges to make the relation symmetric
- By adding those edges, we have made a non-symmetric relation $R$ into a symmetric relation
- This new relation is called the symmetric closure of $R$

Symmetric closure example

- Let $R$ be a relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0,1), (1,1), (1,2), (2,0), (2,2)$, and $(3,0)$
- What is the symmetric closure of $R$?
- We add all pairs of edges $(a,b)$ where $(b,a)$ exists
  - We make all "single" edges into anti-parallel pairs

Symmetric closure example with matrices

- Let $R$ be a relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0,1), (1,1), (1,2), (2,0), (2,2)$, and $(3,0)$
- What is the symmetric closure of $R$?
- We add all pairs of edges $(a,b)$ where $(b,a)$ exists
  - We make all "single" edges into anti-parallel pairs

\[ R = \begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{array} \]

\[ R \cup R^T = \begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{array} = \begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{array} \]

We add edges:
- $(0,2), (0,3), (1,0), (2,1)$
Transitive closure

- Informal definition: If there is a path from \(a\) to \(b\), then there should be an edge from \(a\) to \(b\) in the transitive closure.
- First take of a definition:
  - In order to find the transitive closure of a relation \(R\), we add an edge from \(a\) to \(c\), when there are edges from \(a\) to \(b\) and \(b\) to \(c\).
- But there is a path from 1 to 4 with no edge!

\[ R = \{ (1,2), (2,3), (3,4) \} \]

\[(1,2) \& (2,3) \Rightarrow (1,3) \]
\[(2,3) \& (3,4) \Rightarrow (2,4) \]

Connectivity relation

- \(R\) contains edges between all the nodes reachable via 1 edge.
- \(R \cdot R = R^2\) contains edges between nodes that are reachable via 2 edges in \(R\) (first repeat).
- \(R^2 \cdot R = R^3\) contains edges between nodes that are reachable via 3 edges in \(R\) (second repeat).
- \(R^n\) contains edges between nodes that are reachable via \(n\) edges in \(R\).

- \(R^*\) contains edges between nodes that are reachable via any number of edges (i.e., via any path) in \(R\).
  - Rephrased: \(R^*\) contains all the edges between nodes \(a\) and \(b\) when there is a path of length at least 1 between \(a\) and \(b\) in \(R\).

- \(R^*\) is the transitive closure of \(R\).
  - The definition of a transitive closure is that there are edges between any nodes \((a,b)\) that contain a path between them.

Transitive closure – Matrix Algorithm

\[ R = \{ (1,2), (2,3), (3,4) \} \]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

... it continues on slide 35
Finding the transitive closure with matrices

- Let $M_R$ be the zero-one matrix of the relation $R$ on a set with $n$ elements. Then the zero-one matrix of the transitive closure $R^*$ is:

$$M_{R^*} = M_R \lor M_R^{[2]} \lor M_R^{[3]} \lor \cdots \lor M_R^{[n]}$$

Nodes reachable with one application of the relation
Nodes reachable with two applications of the relation
Nodes reachable with $n$ applications of the relation

Sample questions

- Find the zero-one matrix of the transitive closure of the relation $R$ given by:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \lor M_R^{[2]} \lor M_R^{[3]}$$

$$M_R^{[2]} = M_R \odot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Sample questions

- Now you prove that

$$M_{R^*} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
Transitive closure algorithm

• What we did (or rather, could have done):
  – Compute the next matrix $M_R^{[i]}$, where $1 \leq i \leq n$
  – Do a Boolean join with the previously computed matrix

• For our example:
  – Compute $M_R^{[2]} = M_R \circ M_R$
  – Join that with $M_R$ to yield $M_R \lor M_R^{[2]}$
  – Compute $M_R^{[3]} = M_R^{[2]} \circ M_R$
  – Join that with $M_R \lor M_R^{[2]}$ from above

Transitive closure algorithm

procedure transitive_closure ($M_R$: zero-one $n \times n$ matrix)
A := $M_R$
B := A
for $i := 2$ to $n$
  begin
    A := A $\lor$ $M_R$
    B := B $\lor$ A
  end
{ B is the zero-one matrix for $R^*$ }

More transitive closure algorithms

• More efficient algorithms exist, such as Warshall’s algorithm
  – We won’t be studying it in this class

Equivalence Relations

• Equivalence relations are used to relate objects that are similar in some way.

• Definition: A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

• Two elements that are related by an equivalence relation $R$ are called equivalent.
Equivalence Relations

- Since R is reflexive, every element is equivalent to itself. (For every \( a \in S \), \( aRa \).)
- Since R is symmetric, a is equivalent to b whenever b is equivalent to a. (If \( aRb \) then \( bRa \).)
- Since R is transitive, if a and b are equivalent and b and c are equivalent, then a and c are equivalent. (If \( aRb \) and \( bRc \) then \( aRc \).)

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

The general idea behind an equivalence relation is that it is a classification of objects which are in some way “alike.” In fact, the relation “=” of equality on any set S is an equivalence relation; that is:

1. \( a = a \) for every \( a \in S \).
2. If \( a = b \), then \( b = a \).
3. If \( a = b \) and \( b = c \), then \( a = c \).

More equivalency:

- Consider the set \( L \) of lines in the Euclidean plane. The relation “is parallel to” is an equivalence relation on \( L \).
- The classification of animals by species, that is, the relation “is of the same species as”, is an equivalence relation on the set of animals.
- The relation \( \subseteq \) of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since \( A \subseteq B \) does not imply \( B \subseteq A \).
- Let \( m \) be a fixed positive integer. Two integers \( a \) and \( b \) are said to be congruent modulo \( m \), written

\[ a \equiv b \pmod{m} \]

if \( m \) divides \( a - b \). For example, for \( m = 4 \) we have \( 11 \equiv 3 \pmod{4} \) since 4 divides 11 – 3, and \( 22 \equiv 6 \pmod{4} \) since 4 divides 22 – 6. This relation of congruence modulo \( m \) is an equivalence relation.

Proof that ‘congruence modulo \( m \)’ is an equivalence relation

11 \( \equiv \) 3 (mod 4),

because it's reflexive 11 \( \equiv \) 11 (mod 4),

it's symmetric 3 \( \equiv \) 11 (mod 4),

and it is transitive

11 \( \equiv \) 3 (mod 4) and 3 \( \equiv \) -1 (mod 4),

results into 11 \( \equiv \) -1 (mod 4).

Equivalence Relation

More Examples

- “Strings a and b are the same length.” (see next slide)
- “Integers a and b have the same absolute value.”
- “Integers a and b have the same residue modulo \( m \).” (For a given \( m > 1 \), see previous slide)
Equivalence Relations

• **Example:** Suppose that $R$ is the relation on the set of strings that consist of English letters such that $aRb$ if and only if $L(a) = L(b)$, where $L(x)$ is the length of the string $x$.

  **Is $R$ an equivalence relation?**

  **Solution:**
  • $R$ is reflexive, because $L(a) = L(a)$ and therefore $aRa$ for any string $a$.
  • $R$ is symmetric, because if $L(a) = L(b)$ then $L(b) = L(a)$, so if $aRb$ then $bRa$.
  • $R$ is transitive, because if $L(a) = L(b)$ and $L(b) = L(c)$, then $L(a) = L(c)$, so $aRb$ and $bRc$ implies $aRc$.
  • $R$ is an equivalence relation.

Equivalence Classes

• **Example:** In the previous example (**strings of identical length**), what is the equivalence class of the word mouse, denoted by $[\text{mouse}]$?

  **Solution:** $[\text{mouse}]$ is the set of all English words containing five letters.

  • For example, ‘horse’ would be a representative of this equivalence class.

Theorem: Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

• $aRb$
• $[a] = [b]$
• $[a] \cap [b] \neq \emptyset$

**Definition:** A **partition** of a set $S$ is a collection of disjoint nonempty subsets of $S$ that have $S$ as their union. In other words, the collection of subsets $A_i$, $i \in I$, forms a partition of $S$ if and only if

(i) \( A_i \neq \emptyset \) for $i \in I$
(ii) \( A_i \cap A_j = \emptyset \), if $i \neq j$
(iii) \( \bigcup_{i \in I} A_i = S \)
Examples of partitions

\[ S = \{a, b, M, p, 1, \&\} \]

\begin{itemize}
  \item a \& b
  \item M \& 1
  \item p
  \item YES
  \item NO
\end{itemize}

Equivalence Classes

\textbf{Example:} Let \( S \) be the set \( \{u, m, b, r, o, c, k, s\} \).
Do the following collections of sets partition \( S \) ?

\begin{itemize}
  \item \{\{m, o, c, k\}, \{r, u, b, s\}\} \quad \text{yes.}
  \item \{\{c, o, m, b\}, \{u, s\}, \{r\}\} \quad \text{no (k is missing)}.
  \item \{\{b, r, o, c, k\}, \{m, u, s, t\}\} \quad \text{no (t is not in S)}.
  \item \{\{u, m, b, r, o, c, k, s\}\} \quad \text{yes.}
  \item \{\{b, o, o, k\}, \{r, u, m\}, \{c, s\}\} \quad \text{yes (\{b, o, o, k\} = \{b, o, k\}).}
  \item \{\{u, m, b\}, \{r, o, c, k, s\}, \emptyset\} \quad \text{no (\emptyset not allowed).}
\end{itemize}

Equivalence Classes

\textbf{Theorem:} Let \( R \) be an equivalence relation on a set \( S \).
Then the \textit{equivalence classes} of \( R \) form a \textbf{partition} of \( S \). Conversely, given a partition \( \{A_i | i \in I\} \) of the set \( S \), there is an equivalence relation \( R \) that has the sets \( A_i, i \in I \), as its equivalence classes.

\textbf{Example:} Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Sava lives in Belgrade.

Let \( R \) be the \textbf{equivalence relation} \( \{(a, b) | a \text{ and } b \text{ live in the same city}\} \) on the set \( P = \{\text{Frank, Suzanne, George, Stephanie, Max, Sava}\} \).

Then \( R = \{(\text{Frank, Frank}), (\text{Frank, Suzanne}), (\text{Frank, George}), (\text{Suzanne, Frank}), (\text{Suzanne, Suzanne}), (\text{Suzanne, George}), (\text{George, Frank}), (\text{George, Suzanne}), (\text{George, George}), (\text{Stephanie, Stephanie}), (\text{Stephanie, Max}), (\text{Max, Stephanie}), (\text{Max, Max}), (\text{Sava, Sava})\} \). ... it continues
Equivalence Classes

• Then the equivalence classes of R are:
  • \{\{Frank, Suzanne, George\}, \{Stephanie, Max\}, \{Sava\}\}.
• This is a partition of P.

• The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to exactly one of the equivalence classes.

Equivalence Classes

• Another example: Let R be the relation \((a, b) \mid a \equiv b \pmod{3}\) on the set of integers.
• Is R an equivalence relation?
  • Yes, R is reflexive, symmetric, and transitive.

• What are the equivalence classes of R?
  • \{\{…, -6, -3, 0, 3, 6, …\}, \{…, -5, -2, 1, 4, 7, …\}, \{…, -4, -1, 2, 5, 8, …\}\}

Quick survey

• I understood the material in this slide set...
  a) Very well, or close
  b) With some review, I'll be good
  c) Not really
  d) Not at all

Quick survey

The pace of the lecture for this slide set was...
  a) Fast
  b) About right
  c) A little slow
  d) Too slow
Quick survey

• How interesting was the material in this slide set? Be honest!

a) Wow! That was cooooool!
b) Somewhat interesting
c) Rather boring

d) zzzzzzzzzzzzzzzzzzz