# Perfect $r$-Codes in Strong Products of Graphs

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**Abstract.** A perfect $r$-code in a graph is a subset of the graph's vertices with the property that each vertex in the graph is within distance $r$ of exactly one vertex in the subset. We prove that the $n$-fold strong product of simple graphs has a perfect $r$-code if and only if each factor has a perfect $r$-code.

## 1 Introduction

For a positive integer $r$, a *perfect $r$-code* in a simple graph $G = (V(G), E(G))$ is a subset $C$ of $V(G)$ for which the balls of radius $r$ centered at the vertices of $C$ form a partition of $V(G)$. This idea, introduced in [1], generalizes the notion of a standard error-correcting code. Perfect $r$-codes have also been used to model the problem of efficient placement of resources in a network. If the vertices in the code represent locations of resources, then every vertex in the graph is within distance $r$ of exactly one resource. Aside from applications, the study of perfect $r$-codes is an interesting combinatorial problem unto itself.

Perfect codes appear naturally in products of graphs. Perfect Hamming codes can be understood as perfect $r$-codes in Cartesian products of complete graphs and perfect Lee codes as perfect $r$-codes in Cartesian products of cycles. The most elusive problem in this area concerns perfect codes in the Lee metric. In [4] Golomb and Welch conjectured the nonexistence of $n$-dimensional perfect codes in the Lee metric for $n \geq 3$ and $r \geq 2$. This conjecture was only partially confirmed, see [11, 5, 6, 12] for details. Other results concerning perfect $r$-codes in Cartesian products appear in [2, 3].

Recently a number of authors have studied perfect $r$-codes in direct products [8, 9, 10, 13]. Perfect $r$-codes in a third type of product called the *strong product* have received little or no attention. Our note is a response to this deficiency. We prove constructively that an $n$-fold strong product has a perfect $r$-code if and only if each of its factors has a perfect $r$-code. This result is used to characterize which products of cycles and paths have perfect $r$-codes.

The distance between vertices $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the number of edges in a shortest path from $u$ to $v$. For a vertex $v \in V(G)$, let $B(v, r) = \{u \in V(G) | d_G(u, v) \leq r\}$ denote the *$r$-ball centered at $v$*. Thus, a subset $C \subseteq V(G)$ is a *perfect $r$-code* in $G$ if $\{B(c, r) | c \in C\}$ forms a partition of $V(G)$. If $x \in B(c, r)$, where $c \in C$, we say $x$ is *$r$-dominated by $c$*. In other words, a perfect $r$-code in a simple graph $G$ is a subset $C$ of $V(G)$ such that every vertex of $G$ is $r$-dominated by exactly one vertex in $C$. For example, in Figure 1, the dark vertices form a perfect 3-code. Each vertex is 3-dominated by exactly one member of the code. The 3-balls centered at the dark vertices are indicated by dotted lines. It is a simple matter to show that any two perfect $r$-codes in a given graph have the same cardinality.

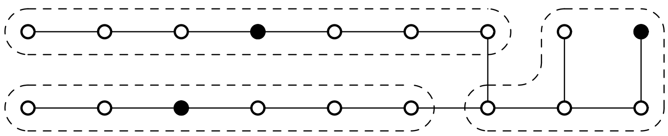

Figure 1

The *strong product* of graphs $G$ and $H$ is the graph $G \boxtimes H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are the pairs $(g, h)(g', h')$ of distinct vertices for which one of the following holds:

1. $g = g'$ and $hh' \in E(H)$
2. $gg' \in E(G)$ and $h = h'$
3. $gg' \in E(G)$ and $hh' \in E(H)$.

The graphs $G$ and $H$ are called *factors* of the product. The strong product also appears in literature as the *strong direct product* or *symmetric composition*. As an example of a strong product, Figure 2 shows $P_3 \boxtimes C_4$ where $P_n$ denotes the path on $n$ vertices and $C_n$ is a cycle on $n$ vertices.

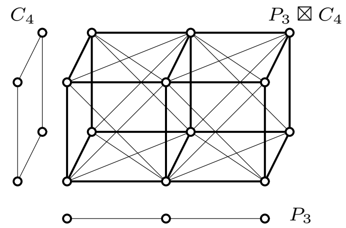


Figure 2

For clarity, the edges of form (1) and (2) are displayed bold. We note that edges of these types form what is called the *Cartesian* product of $G$ and $H$. The edges of form (3) yield the *direct* or *tensor* product of $G$ and $H$. (See [7] for details.) Thus the edges of $G \boxtimes H$ are the union of the edges of the Cartesian and direct products.

The strong product is associative in the sense that the map $(g_1, (g_2, g_3)) \mapsto ((g_1, g_2), g_3)$ is an isomorphism from $G_1 \boxtimes (G_2 \boxtimes G_3)$ to $(G_1 \boxtimes G_2) \boxtimes G_3$. Thus $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_2$ is well-definded without regard to grouping of factors, so it is natural to drop the parentheses. Doing this leads to the following fact which we accept as the definition of an $n$-fold strong product. The *$n$-fold strong product* $\boxtimes_{i=1}^{n} G_i = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ consists of vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$ where a pair $(g_1, g_2, \ldots, g_n)(g'_1, g'_2, \ldots, g'_n)$ of distinct vertices is an edge exactly when $g_i = g'_i$ or $g_i g'_i \in E(G_i)$ for each $1 \leq i \leq n$.

By [7, Lemma 2.1] the distance between two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in the graph $G = \boxtimes_{i=1}^{n} G_i$ is

$$d_G(u, v) = \max_{1 \leq i \leq n} d_{G_i}(u_i, v_i). \tag{1}$$

In what follows $\pi_i$ denotes the usual projection functions, $\pi_i : V(\boxtimes_{i=1}^{n} G_i) \to V(G_i)$ defined by $\pi_i(g_1, g_2, \ldots, g_n) = g_i$. For more details on the strong product see [7].

## 2 Results

In this section we examine the relationship between perfect $r$-codes in the $n$-fold strong product of graphs and perfect $r$-codes in their factors. We show that an $n$-fold strong product of graphs has a perfect $r$-code if and only if each factor has a perfect $r$-code. We start by proving the converse.

**Proposition 2.1** *Suppose $G_1, G_2, \ldots, G_n$ are graphs and $G_i$ has a perfect $r$-code $C_i \subseteq V(G_i)$ for $1 \leq i \leq n$. Then $C_1 \times C_2 \times \cdots \times C_n$ is a perfect $r$-code in $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$.*

*Proof.* Set $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. Suppose that $C_i \subseteq V(G_i)$ is a perfect $r$-code in $G_i$ for $1 \leq i \leq n$. Form the Cartesian product $C = C_1 \times C_2 \times \cdots \times C_n$. We claim that $C$ is a perfect $r$-code in $G$.

Let $g = (g_1, g_2, \ldots, g_n) \in V(G)$. Then each $g_i \in V(G_i)$ is within a distance $r$ of some $c_i \in C_i$. Let $c = (c_1, c_2, \ldots, c_n)$. Then by (1), $d_G(g, c) = \max_{1 \leq i \leq n} d_{G_i}(g_i, c_i) \leq r$. Hence every vertex in $G$ is $r$-dominated by a vertex in $C_1 \times C_2 \times \cdots \times C_n$.

Now suppose there exists some $g = (g_1, g_2, \ldots, g_n) \in V(G)$ that is $r$-dominated by vertices $c = (c_1, c_2, \ldots, c_n)$ and $c' = (c'_1, c'_2, \ldots, c'_n)$ in $C$. Then $d_G(g, c) = \max\{d_{G_i}(g_i, c_i)\} \leq r$ and $d_G(g, c') = \max\{d_{G_i}(g_i, c'_i)\} \leq r$. Thus for each $1 \leq i \leq n$, we have $d_{G_i}(g_i, c_i) \leq r$ and $d_{G_i}(g_i, c'_i) \leq r$ for code elements $c_i, c'_i \in C_i$. Hence $c_i = c'_i$, so $c = c'$. Thus $C$ is a perfect $r$-code in $G$. ∎

Figure 3a illustrates Proposition 2.1 where the dark vertices belong to perfect 2-codes in the factors and product. Indeed, the Cartesian product of the codes in the factors is a code in the product. However, as illustrated in Figure 3b, not every code in a product is a Cartesian product of codes in the factors. Thus a simple projection of a code in the product to a factor will not always produce a code in the factor. The next proposition shows how to construct codes in the factors from a code in the product.

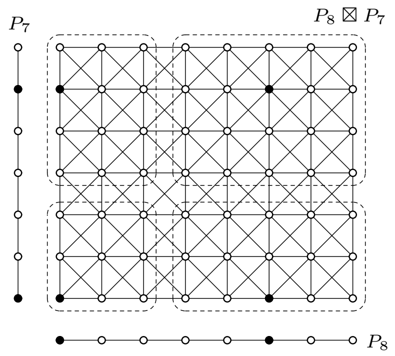


Figure 3a

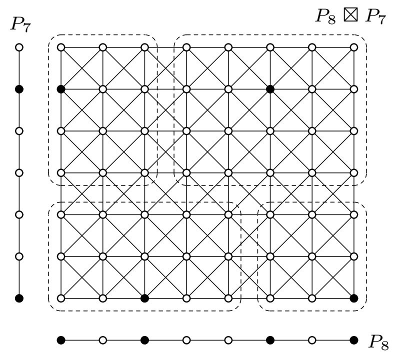


Figure 3b

**Proposition 2.2** *Let $G_1, G_2, \ldots, G_n$ be graphs and let $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. Suppose that $C$ is a perfect $r$-code in $G$ and fix $(g_1, \ldots, g_n) \in V(G)$.*

*For $1 \leq i \leq n$, set $D_i = \{(x_1, \ldots, x_n) \in V(G) | d_{G_j}(g_j, x_j) \leq r$ for $j \neq i\}$. Then $C_i = \pi_i(C \cap D_i)$ is a perfect $r$-code in $G_i$ for each $1 \leq i \leq n$.*

*Proof.* Suppose $C$ is a perfect $r$-code in $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ and let $v$ be any vertex in $G_i$. Since $G$ has perfect $r$-code $C$, the vertex $(g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n)$ in $G$ must be $r$-dominated by some $(c_1, c_2, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_n) \in C$. Thus

$$d_G((g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n), (c_1, c_2, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_n)) \leq r,$$

which by (1) implies that

$$\max\{d_{G_1}(g_1, c_1), d_{G_2}(g_2, c_2), \ldots, d_{G_i}(v, c_i), \ldots, d_{G_n}(g_n, c_n)\} \leq r. \qquad (2)$$

Thus, $d_{G_j}(g_j, c_j) \leq r$ for each $j \neq i$ so $(c_1, \ldots, c_i, \ldots, c_n) \in C \cap D_i$. Hence $c_i \in \pi_i(C \cap D_i) = C_i$. But (2) also implies $d_{G_i}(v, c_i) \leq r$, hence $v$ is $r$-dominated by $c_i \in C_i$.

Now suppose that $v$ is $r$-dominated by two elements $c_i$ and $c'_i$ in $C_i$. Then $c_i = \pi_i((c_1, c_2, \ldots, c_n))$ where $(c_1, c_2, \ldots, c_n) \in C \cap D_i$ and $c'_i = \pi_i((c'_1, c'_2, \ldots, c'_n))$ where $(c'_1, c'_2, \ldots, c'_n) \in C \cap D_i$. Thus, by definition of $D_i$ we have, $d_{G_j}(g_j, c_j) \leq r$ and $d_{G_j}(g_j, c'_j) \leq r$ for $j \neq i$. By assumption, $d_{G_i}(v, c_i) \leq r$ and $d_{G_i}(v, c'_i) \leq r$. Therefore by (1),

$$d_G((g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n), (c_1, c_2, \ldots, c_n)) \leq r$$

and

$$d_G((g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n), (c'_1, c'_2, \ldots, c'_n)) \leq r.$$

This implies that $(c_1, \ldots, c_n) = (c'_1, \ldots, c'_n)$, in particular $c_i = c'_i$. Thus $C_i$ is a perfect $r$-code in $G_i$ for every $1 \leq i \leq n$. ■

**Theorem 2.1** *Suppose $G_1, G_2, \ldots, G_n$ are graphs and let $G = G_1 \boxtimes G_2 \boxtimes \ldots \boxtimes G_n$. Then $G$ has more than one perfect $r$-code if and only if each factor $G_i$ for $1 \leq i \leq n$ has at least one perfect $r$-code and one factor has more than one perfect $r$-code.*

*Proof.* Suppose that $G$ has perfect $r$-codes $C$ and $C'$. Then by Proposition 2.2, $G_i$ has at least one perfect $r$-code for each $1 \leq i \leq n$. We show now that $G_k$, for some $1 \leq k \leq n$, has two perfect $r$-codes. Since $G$ has two perfect $r$-codes there must exist a vertex $(g_1, \ldots, g_n)$ in $G$ that is $r$-dominated by $(c_1, \ldots, c_n) \in C$ and $(c'_1, \ldots, c'_n) \in C'$ where $c_k \neq c'_k$ for some $1 \leq k \leq n$. By Proposition 2.2, $C_k = \pi_k(C \cap D_k)$ and $C'_k = \pi_k(C \cap D'_k)$ are perfect $r$-codes in $G_k$. Notice that $C_k$ and $C'_k$ are not equal. Clearly $c_k \in C_k$ and $c'_k \in C'_k$, however, $c_k \notin C'_k$ for this would imply that $g_k \in G_k$ would be $r$-dominated by $c_k$ and $c'_k$ in $C'_k$. Thus, $C_k$ and $C'_k$ are distinct $r$-codes in $G_k$.

Conversely, suppose that $G_i$ has perfect $r$-code $C_i$ for each $1 \leq i \leq n$. Suppose also that $C_1'$ is another perfect $r$-code in $G_1$. Then by Proposition 2.1, $C_1 \times C_2 \times \cdots \times C_n$ and $C_1' \times C_2 \times \cdots \times C_n$ are both perfect $r$-codes in $G$. ■

Although we have the above theorem, Figures 4a and 4b illustrate that it is not possible to determine the number of perfect $r$-codes in a strong product based on the number of perfect $r$-codes in the factors. The graph $G$ admits two perfect 2-codes and the graphs $H$ and $K$ both admit three perfect 2-codes. However, $H \boxtimes G$ has six perfect 2-codes formed by the Cartesian product of the codes in the factors while $K \boxtimes G$ has twelve perfect 2-codes, six coming from the Cartesian products of the codes in the factors and six more that contain vertices in a staggered pattern. The dark vertices in $K \boxtimes G$ indicate one of the staggered perfect 2-codes.

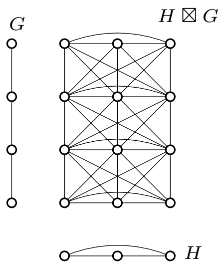


Figure 4a

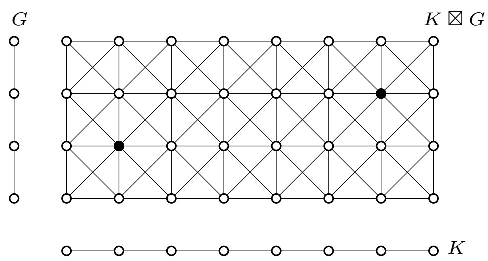


Figure 4b

In a series of papers [8, 9, 10, 13], Jerebic, Jha, Klavžar, Špacapan and Žerovnik characterize the conditions under which a direct product of cycles admits a perfect $r$-code. The situation is remarkably complex. By contrast, our propositions show the analogous problem for the strong product is quite simple. In fact, we can state a result not just for the strong product of cycles, but paths as well. It is simple to check that, for a given $r$, the cycle $Z_s$ on $s$ vertices admits a perfect $r$-code if and only $s$ is a multiple of $2r + 1$, and that any path $P_t$ admits a perfect $r$-code no matter the value of $t$.

**Corollary 2.1** *A product* $(\boxtimes_{i=1}^{m} Z_{s_i}) \boxtimes (\boxtimes_{i=1}^{n} P_{t_i})$ *admits a perfect* $r$*-code if and only if each* $s_i$ *is a multiple of* $2r + 1$.

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