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CIRCULARITY OF PLANAR GRAPHS

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Abstract. A *circular cover* of a graph G is a cover $\{X_0, \cdots, X_{n-1}\}$ of the topological space G by closed connected subsets, indexed over $\mathbb{Z}_n$, with the following properties: Each element in the cover contains a vertex of G, each vertex of G is contained in at most two elements of the cover, and $X_a \cap X_b \neq \emptyset$ if and only if $b - a \in \{-1, 0, 1\}$. The *circularity* of G is the largest integer n for which there is a circular cover of G with n elements. It is known that the circularity of a planar graph is even. We sharpen this result by proving that the circularity of a plane graph is twice the maximum number of disjoint paths joining two faces of G. This result leads to a polynomial-time algorithm which computes the circularity of any connected planar graph.

1. Introduction

A *graph* G is a finite vertex set $V(G)$ together with an edge set $E(G)$ composed of two-element subsets of $V(G)$. An edge $\{v, w\} \in E(G)$ is abbreviated vw (or wv). In addition to this usual combinatorial interpretation, we regard a graph as a topological space; its vertices are points in Euclidean space $\mathbb{R}^n$, each edge vw is a simple arc joining v to w, and edges intersect only at vertices. A *circular cover* of a graph G is a finite cover $\{X_0, \cdots, X_{n-1}\}$ of the space G by closed connected subsets, indexed over the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, and satisfying the following properties: Each X_a contains at least one vertex of G, each vertex of G is in at most two of the X_a's, and $X_a \cap X_b \neq \emptyset$ if and only if $b - a \in \{-1, 0, 1\}$. The *circularity* of G is defined to be the integer

$$\sigma(G) = \max \{n \mid G \text{ has a circular cover with } n \text{ elements }\}. \tag{1.1}$$

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For example, Figure 1 displays a circular cover of C_4 (the four-cycle) by 8 elements, so $\sigma(C_4) \geq 8$. In fact, since any of the four vertices can be in at most 2 elements of a circular cover, it follows that $\sigma(C_4) \leq 8$. Hence $\sigma(C_4) = 8$. Similarly $\sigma(C_p) = 2p$ for any integer $p \geq 3$.

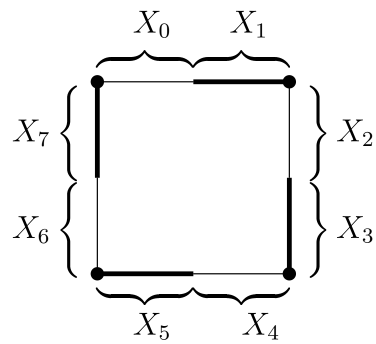

FIGURE 1

Very little is known about the circularity of arbitrary graphs. This paper explores the circularity of the class of planar graphs. Our main result (Theorem 3.3) is that the circularity of a two-connected planar graph is twice the maximum number of disjoint paths joining two faces of a planar embedding. Further, we describe how this leads to a polynomial-time algorithm which computes the circularity of such a graph. We begin with a survey of results from the literature. The notion of an admissible map (introduced in [1]) is then reviewed in Section 2, and this is employed in the proof of our main result in Section 3.

The circularity of several classes of graphs is studied in [1]. It is proved there that the circularity of the complete graphs K_p and the complete bipartite graphs $K(p,q)$ is 6 when $p \geq 3, q \geq 2$ (Proposition 4.2 and Theorem 4.5). Furthermore, it is shown (Theorem 5.4) that for $p \geq 6$ the circulant graph $C_p\langle 1, 2\rangle$ has circularity p, so for any integer $p \geq 6$ there is a graph with circularity p. By Theorem 4.4 of [2], the circularity of a planar graph is even. This becomes a corollary of our main result.

It is also proved in [2] that $\sigma(G) = 2$ if and only if G is a tree, and $\sigma(G) \geq 6$ if and only if G is a connected graph which contains a cycle (Theorem 2.2 and Corollary 2.3). Moreover, it is proved (Theorem 3.6) that if G is connected then $\sigma(G) = \max\{\sigma(B_i) | 1 \leq i \leq k\}$, where $\{B_1, \cdots, B_k\}$ is the set of blocks of G. Now, either a bock is isomorphic to K_2 or it is two-connected. Since $\sigma(K_2) = 2$, the computation of the circularity of an arbitrary connected graph can thus be reduced to computation of the circularity of two-connected graphs. Therefore, the condition of two-connectivity in our main result does not restrict its generality. Notice that since any two-connected graph is connected and contains a cycle, its circularity is at least 6.

We adopt the lexicon and notation of [6]. A graph is *planar* if it can be embedded in the Euclidean plane $\mathbb{R}^2$, and a planar graph with a fixed embedding in $\mathbb{R}^2$ is called a *plane* graph. A plane graph G is regarded as a subspace of $\mathbb{R}^2$; its vertex set is a finite set of points in $\mathbb{R}^2$, and its edges are closed arcs joining pairs of vertices. (However, if X is a subgraph of G, then $G-X$ denotes the graph obtained by removing from G all of X and the edges incident with it – not the space G with the points X removed.) The connected components of $\mathbb{R}^2 - G$ are called the *faces* of G, and the set of faces is denoted $F(G)$. The topological closure of a face Y is denoted $\overline{Y}$, and the *boundary* of Y is $\partial Y = \overline{Y} - Y$, which is a subgraph of G. If G is two-connected then ∂Y is a cycle (cf. Proposition 4.2.5 of [6]). The sets $V(\partial Y)$ and $E(\partial Y)$ are abbreviated $V(Y)$ and $E(Y)$, respectively. If A and B are disjoint subgraphs of G, then $E(A,B) = \{vw \in E(G) \mid v \in V(A), w \in V(B)\}$. The cardinality of a set S is denoted $|S|$.

The *edge space* $\mathcal{E}(G)$ of G is the power set of $E(G)$ endowed with the structure of a vector space over the two-element field $\mathbb{F}_2$. Addition is symmetric difference of sets and zero is the empty set. There is a bilinear form $(\ ,\ )$ on $\mathcal{E}(G)$ defined by declaring (α, β) to be 0 or 1 depending on whether $|\alpha \cap \beta|$ is even or odd. The *cycle space* $\mathcal{C}(G)$ of G is the subspace of $\mathcal{E}(G)$ spanned by the edge sets of the cycles of G.

2. Admissible Maps

The idea of an admissible map, introduced in [1], puts the notion of a circular cover into a combinatorial setting, thus simplifying our proofs. To motivate this definition, observe that any circular cover $\mathcal{X} = \{X_0, \cdots, X_{n-1}\}$ of a graph G induces a map $f^{\mathcal{X}} : V(G) \to \mathbb{Z}_n \times \mathbb{Z}_n$ defined by

$$f^{\mathcal{X}}(v) = \begin{cases} (a, a+1) & \text{if } v \in X_a \cap X_{a+1} \\ (a, a) & \text{if } v \notin X_b \text{ for } a \neq b. \end{cases}$$

The fact that it comes from a circular cover gives $f^{\mathcal{X}}$ a certain structure. We next recall from [1] some notations that aid in formalizing this structure. This notation will be employed throughout our paper.

Let $A(n) = \{(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n \mid b - a \in \{0,1\}\}$, and for each $a \in \mathbb{Z}_n$ define $T(a) = \{(a-1,a),(a,a),(a,a+1)\} \subseteq A(n)$. Given a map $f : V(G) \to A(n)$, and $a \in \mathbb{Z}_n$, let $G(a)$ be the subgraph of G induced by the vertices $f^{-1}(T(a))$. (If $f = f^{\mathcal{X}}$, then $G(a)$ is the subgraph of G induced by the vertices in X_a.) The next definition characterizes those maps $f^{\mathcal{X}}$ coming from circular covers.

Definition 1. A map $f : V(G) \to A(n)$ is said to be *admissible* if each of the following conditions hold:

(a) If $vw \in E(G)$, then $v, w \in V(G(a)) \cup V(G(a+1))$ for some $a \in \mathbb{Z}_n$.
(b) If $a \in \mathbb{Z}_n$, then $G(a)$ is connected.
(c) If $a \in \mathbb{Z}_n$, then either $G(a) \cap G(a+1) \neq \emptyset$ or $E(G(a), G(a+1)) \neq \emptyset$.

The connection between admissible maps and circular covers is given in the next proposition, an immediate consequence of Theorem 2.5 of [1].

Proposition 2.1. *Let $n \geq 6$. For each admissible map $f : V(G) \to A(n)$, there is a circular cover of G by n elements. For each circular cover of G by $n \geq 6$ elements, there is an admissible map $f : V(G) \to A(n)$.*

This proposition allows us to rephrase Definition (1.1) of circularity. If G is two-connected its circularity is at least 6 and Proposition 2.1 gives

$$\sigma(G) = \max \{n \mid \text{there is an admissible map } f : V(G) \to A(n)\}. \tag{2.1}$$

Thus, circularity is a purely graph-theoretic invariant, and need not be formulated in terms the underlying topology of a graph. For the rest of this paper we will use the characterization (2.1) of circularity of two-connected graphs.

If G is a plane graph and $f : V(G) \to A(n)$ is admissible, then a face $Y \in F(G)$ is said to be *saturated* if $V(Y) \cap V(G(a)) \neq \emptyset$ for every $a \in \mathbb{Z}_n$. The existence of saturated faces in a plane graph is an important step towards our results on circularity. Versions of the next proposition have been proved in [2] and [3].

Proposition 2.2. *Suppose G is a two-connected plane graph, $f : V(G) \to A(n)$ is admissible, and $n \geq 6$. Then G has two saturated faces.*

Proof. Let G and f be as stated in the hypothesis. By Theorem 5.4 of [4], the set $\mathcal{W} = \{E(W) | W \in F(G)\}$ spans the cycle space $\mathcal{C}(G)$. In what follows we construct a linear map $d : \mathcal{C}(G) \to \mathbb{F}_2$ having the property that a face Y is saturated if $d(E(Y)) = 1$. The proposition will then be proved by showing that d is nonzero on two elements of $\mathcal{W}$.

Form the set $\Upsilon = E\left(G(0), (G(1) - G(0)) \cup G(2)\right)$ and define d as $d(\lambda) = (\lambda, \Upsilon)$. In other words, $d(\lambda)$ equals 0 or 1 depending on whether $|\lambda \cap \Upsilon|$ is even or odd.

We claim that a face Y is saturated if $d(E(Y)) = 1$. To prove this, we suppose Y is not saturated, and show $d(E(Y)) = 0$. Since Y is not saturated, there is an $a \in \mathbb{Z}_n$ for which $V(Y) \cap V(G(a)) = \emptyset$. Consider the subgraph $A = \partial Y \cap (G(a+1) \cup G(a+2) \cup \cdots \cup G(0))$ of the cycle ∂Y, which by condition (a) of Definition 1 has no edges in Υ. By construction, if $e \in E(Y) \cap \Upsilon$, one endpoint of e is in $V(A)$, and by condition (a) of Definition 1 the other endpoint is not in $V(A)$. Moreover, the component of A with which e is incident must be a path (since A is a proper subgraph of the cycle ∂Y), and condition (a) implies that this path joins e to another edge of $E(Y) \cap \Upsilon$. It follows that $E(Y) \cap \Upsilon$ can be partitioned into pairs of edges, each pair joined by a component of A. Thus $|E(Y) \cap \Upsilon|$ is even, so $d(E(Y)) = 0$.

To verify that d is nonzero, we construct a cycle on which d is 1. Take the subgraphs $H = G(0) \cup G(1) \cup G(2)$, $J = G(2) \cup G(3) \cup G(4)$, and $K = G(4) \cup G(5) \cup \cdots \cup G(0)$ of G, which are connected by conditions (b) and (c) of Definition 1. Notice that $E(J) \cap \Upsilon = \emptyset = E(K) \cap \Upsilon$, while $\Upsilon \subseteq E(H)$. Choose $v, w \in V(H)$ with $v \in V(G(0))$ and $w \in V(G(2))$, and let P be a v-w path in H. Now, P necessarily contains edges in Υ; let Q be the shortest x-w subpath of P whose first edge is in in Υ. Then $x \in V(G(0))$, $w \in V(G(2))$ and $|E(Q) \cap \Upsilon| = 1$. Now take $y \in V(G(4))$, let R be a w-y path in J, and let S be a y-x path in K. Since $(E(R) \cup E(S)) \cap \Upsilon = \emptyset$, it follows that the cycle QRS intersects Υ at a single edge. Consequently $d(E(QRS)) = 1$, so d is nonzero.

Finally, to complete the proof we show there are two faces $Y, Z \in F(G)$ for which $d(E(Y)) = 1 = d(E(Z))$. Certainly, since $\mathcal{W} = \{E(W) | W \in F(G)\}$ spans $\mathcal{C}(G)$ and d is nonzero, there must be some $Y \in F(G)$ for which $d(E(Y)) = 1$. Now, any edge of G belongs to exactly two faces, so it follows that $0 = \sum_{Z \in F(G)} E(Z)$, or rather $E(Y) = \sum_{Z \in F(G) - \{Y\}} E(Z)$. Taking d of both sides, $1 = \sum_{Z \in F(G) - \{Y\}} d(E(Z))$, so $d(E(Z)) = 1$ for some face $Z \neq Y$. This means faces Y and Z are saturated. $\square$

3. Main Results

In this section G is assumed to be a two-connected plane graph. As mentioned in the introduction, such a graph necessarily has circularity at least 6, and all its faces are bounded by cycles. The following construction will be useful in our proofs.

Given two faces Y and Z of G, a new plane graph G_{YZ} is formed from G as follows: Add two new vertices y and z inside the faces Y and Z respectively, so $V(G_{YZ}) = V(G) \cup \{y, z\}$. Put $E(G_{YZ}) = E(G) \cup \{yx | x \in V(Y)\} \cup \{zx | x \in$

$V(Z)\}$, and embed the new edges inside the faces Y and Z in such a way that no edges cross. Then G is a subgraph of the plane graph G_{YZ}.

We say two faces Y and Z of G are connected by n disjoint paths if there are n paths in G, with pairwise disjoint vertex sets, each joining a vertex of Y to a vertex of Z (we allow for the possibility that such a path consists only of a single vertex, as may happen if Y and Z share a vertex). This is equivalent to saying G_{YZ} has n internally disjoint y-z paths.

The next two lemmas – relating admissible maps to disjoint paths joining faces of G – are the primary technical results of this paper.

Lemma 3.1. *Suppose $f : V(G) \to A(n)$ is admissible, and Y and Z are saturated faces of G. Then there are $\lceil n/2 \rceil$ disjoint paths joining Y to Z.*

Proof. Let G, f, Y, and Z be as in the statement of the lemma, let $k = \lceil n/2 \rceil$, and let G_{YZ} be as described above. The lemma will be proved if it can be shown that removal of fewer than k vertices of G cannot disconnect the vertices y and z of G_{YZ}, for then Menger's theorem (cf. Theorem 3.3.1 of [6]) asserts there are at least k internally disjoint y-z paths in G_{YZ} (and hence at least $k = \lceil n/2 \rceil$ disjoint paths joining Y and Z).

Thus, let $v_1, \cdots, v_{k-1}$ be $k-1$ vertices of G. We will produce a y-z path in G_{YZ} which misses all of these vertices. Define the set $S = \{a \in \mathbb{Z}_n | v_i \in V(G(a)), 1 \leq i \leq k-1\}$. By definition of $G(a)$, a vertex $v_i \in \{v_1, \cdots, v_{k-1}\}$ can belong to at most two of the $G(a)$, so S has cardinality no greater than $2(k-1) < n$. Consequently, there is an element $a_0 \in \mathbb{Z}_n$ that is not in S. Since Y and Z are saturated faces, there are vertices $y_0 \in V(Y)$ and $z_0 \in V(Z)$ that are vertices of $G(a_0)$. By condition (b) of Definition 1, there is a path P in $G(a_0)$ joining y_0 to z_0. Since $v_i \notin V(G(a_0))$ for $1 \leq i \leq k-1$, it follows that the path $yy_0 \cup P \cup z_0 z$ joins y to z and misses every one of the vertices $v_1, \cdots, v_{k-1}$. □

Lemma 3.2. *If there are n disjoint paths joining faces Y and Z of G, then there is an admissible map $f : V(G) \to A(2n)$.*

Proof. Suppose there are n disjoint paths joining Y to Z, so there are n internally disjoint y-z paths in G_{YZ}. Observe that the subgroup $I = \langle 2 \rangle = \{0, 2, 4, \cdots, 2n-2\}$ of $\mathbb{Z}_{2n}$ has exactly n elements, so we index our internally disjoint y-z paths $P_0, P_2, P_4, \cdots, P_{2n-2}$ over this subgroup I of "even" elements of $\mathbb{Z}_{2n}$. Moreover, we assume the indexing to be in clockwise order around the vertex y, so that $\mathbb{R}^2 - \bigcup_{a \in I} P_a$ has n connected components $R_0, R_2, R_4, \cdots, R_{2n-2}$, with each R_a bounded by $P_a \cup P_{a+2}$. Put $R_a^+ = P_a \cup R_a \subseteq \mathbb{R}^2$, so any vertex of G is in exactly one of the n sets R_a^+, $a \in I$. (See Figure 2.)

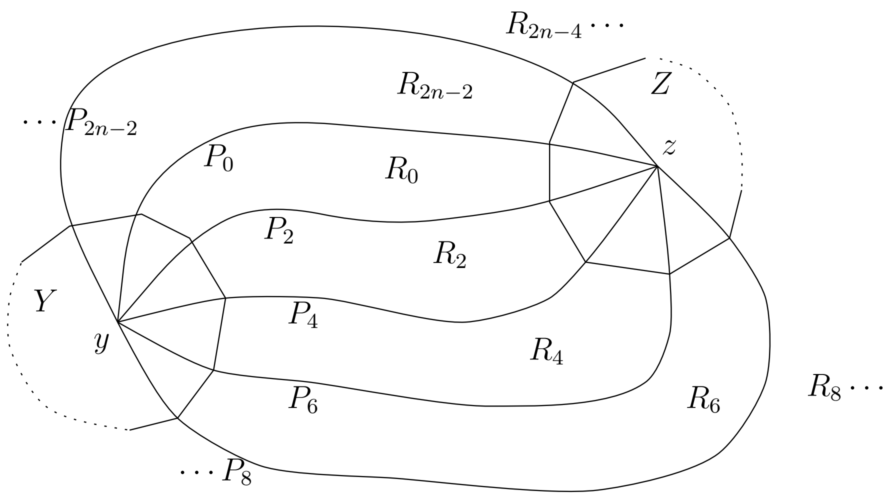

FIGURE 2

For each $a \in I$, let H_a be the subgraph of G induced by the vertices in R_a^+, let A_a be the connected component of H_a which contains $P_a - \{y, z\}$, and let B_a be the union (possibly empty) of the remaining components of H_a. Observe that the sets $\{V(A_a), V(B_a) | a \in I\}$ form a partition of $V(G)$.

We define the function $f : V(G) \to A(2n)$ as follows.

$$f(v) = \begin{cases} (a, a+1) & \text{if } v \in V(A_a) \\ (a+2, a+3) & \text{if } v \in V(B_a) \end{cases}$$

To finish the proof we show that f satisfies conditions (a), (b), and (c) in the definition (1) of an admissible map.

Notice that f has been defined so that if $a \in I$, then $f^{-1}(T(a)) = f^{-1}(a, a+1) = f^{-1}(T(a+1))$. It follows that, if $a \in I$, then $G(a) = G(a+1)$ and $V(G(a)) = V(G(a+1)) = f^{-1}(a, a+1) = V(A_a) \cup V(B_{a-2})$.

If $vw \in E(G)$, then $v, w \in R_a^+ \cup P_{a+2}$ for some $a \in I$. It follows that $v, w \in V(H_a) \cup V(A_{a+2}) = V(A_a) \cup V(B_a) \cup V(A_{a+2}) \subseteq V(G(a)) \cup V(G(a+2)) = V(G(a+1)) \cup V(G(a+2))$. This verifies that f satisfies condition (a).

To verify condition (b), we must show that $G(a)$ is connected for each $a \in \mathbb{Z}_n$. Since $G(a+1) = G(a)$ for $a \in I$, we need only verify $G(a)$ is connected when $a \in I$. So let $a \in I$. Since $V(G(a)) = V(A_a) \cup V(B_{a-2})$, and A_a is connected by definition, it suffices to show that any component C of B_{a-2} has a vertex adjacent to A_a. By connectivity of G, $E(C, G-C)$ contains an edge vw. If $v \in V(C)$, then $v \in R_{a-2}$, and $w \in R^+_{a-2} \cup P_a$, so $w \in V(H_{a-2}) \cup V(P_a - \{y,z\}) = V(A_{a-2}) \cup V(B_{a-2}) \cup V(P_a - \{y,z\})$. It is impossible for w to be a vertex of A_{a-2}, for otherwise C would be a part of A_{a-2}. Similarly, $w \notin V(B_{a-2})$, for otherwise $w \in V(C)$, contrary to assumption. It follows that $w \in V(P_a - \{y,z\}) \subseteq V(A_a)$. Thus f satisfies condition (b).

Finally, to demonstrate that f satisfies condition (c), let $a \in \mathbb{Z}_n$. If $a \in I$ then any vertex $v \in V(P_a) - \{y,z\}$ is in $V(G(a)) = V(G(a) \cap G(a+1))$. If $a \notin I$, then, by definition of f, the path $P = \partial Y \cap (R^+_{a-1} \cup P_{a+1})$ has the property that $f(V(P)) = \{(a-1,a),(a+1,a+2)\}$, so there is some $vw \in E(P)$ with $f(v) = (a-1,a)$ and $f(w) = (a+1,a+2)$. Consequently, $vw \in E(G(a-1), G(a+1)) = E(G(a), G(a+1))$. This verifies that f satisfies condition (c) and also completes the proof of the lemma. $\square$

Given two faces Y and Z of G, let $M(Y,Z)$ denote the maximum number of disjoint paths in G joining Y to Z, or, what is the same, the maximum number of internally disjoint y-z paths in G_{YZ}. Let $\kappa(Y,Z)$ be the least number of vertices of G whose removal disconnects y and z, so $M(Y,Z) = \kappa(Y,Z)$ by Menger's theorem. The next theorem expresses the circularity of a plane graph in terms of the functions M or κ.

Theorem 3.3. *If G is a two-connected plane graph, then $\sigma(G) = \max\{2M(Y,Z) | Y,Z \in F(G)\} = \max\{2\kappa(Y,Z) | Y,Z \in F(G)\}$.*

Proof. Any two faces Y and Z of G are joined by $M(Y,Z)$ disjoint paths in G. By Lemma 3.2, there is an admissible map $f : V(G) \to A(2M(Y,Z))$, so $\sigma(G) \geq 2M(Y,Z)$. Hence $\sigma(G) \geq \max\{2M(Y,Z) | Y,Z \in F(G)\}$.

To establish the reverse inequality, choose an admissible map $f : V(G) \to A(\sigma(G))$. Now, $\sigma(G) \geq 6$, so Proposition 2.2 guarantees that G has two saturated faces Y_0 and Z_0. Using Lemma 3.1, $\sigma(G) \leq 2\left\lceil \frac{\sigma(G)}{2} \right\rceil \leq 2M(Y_0, Z_0) \leq \max\{2M(Y,Z) | Y,Z \in F(G)\}$. $\square$

The next result has been known for some time (cf. Theorem 4.4 of [2]), but is included here because it is an immediate corollary of the previous theorem.

Corollary 3.4. *If G is a connected planar graph, then $\sigma(G)$ is even.*

Theorem 3.3 suggests an algorithm for computing the circularity of an arbitrary two-connected plane graph G. According to this theorem, $\sigma(G)$ is twice the maximum of the numbers $M(Y,Z)$, where Y and Z range over all pairs of distinct faces of a plane embedding of G. Now, the number $M(Y,Z)$ equals the value of a maximal flow in a certain network N associated to the graph G_{YZ}; for details, the reader is referred to the proof of Theorem 5.9 of [5]. A maximal flow in N can be computed using, say, the Edmonds-Karp max-flow min-cut algorithm (Algorithm 5.1 of [5]). It is proved in section 5.4 of [5] that the complexity of using this method to find the flow is $O(pq^2)$, where G has p vertices and q edges.

The circularity of a two-connected plane graph G can thus be computed by calculating $2M(Y,Z)$ for every pair $Y,Z \in F(G)$ and selecting the largest value thus obtained. Each computation of $2M(Y,Z)$ has complexity $O(pq^2)$, and if G has r faces this computation must be made $r(r-1)/2$ times. Using Euler's formula, $r(r-1)/2 < r^2 = (q-p+2)^2 < q^2$, so the complexity of using this method to compute $\sigma(G)$ is $O(pq^4)$.

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