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DIRECT PRODUCT FACTORIZATION OF BIPARTITE GRAPHS WITH BIPARTITION-REVERSING INVOLUTIONS*

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Abstract. Given a connected bipartite graph G, we describe a procedure which enumerates and computes all graphs H (if any) for which there is a direct product factorization $G \cong H \times K_2$. We apply this technique to the problems of factoring even cycles and hypercubes over the direct product. In the case of hypercubes, our work expands some known results by Brešar, Imrich, Klavžar, Rall, and Zmazek [*Finite and infinite hypercubes as direct products*, Australas. J. Combin., 36 (2006), pp. 83–90, and *Hypercubes as direct products*, SIAM J. Discrete Math., 18 (2005), pp. 778–786].

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1. Introduction. A graph $G = (V(G), E(G))$ in this paper is finite and may have loops, but not multiple edges. The *direct product* of two graphs G and H is the graph $G \times H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are $E(G \times H) = \{(g, h)(g', h') : gg' \in E(G), hh' \in E(H)\}$. The graphs G and H are called *factors* of the product.

If I is the graph with one vertex and one loop, then $I \times G \cong G$ for any graph G. A graph G is said to be *prime* with respect to $\times$ if whenever $G \cong H \times K$, one factor is isomorphic to I and the other is isomorphic to G. A fundamental result due to McKenzie [7] (see also Imrich [6]) implies that any connected non-bipartite graph has a unique prime factorization over the direct product. It is known that bipartite graphs are not uniquely prime factorable, but the ways that they can decompose into prime factors is largely unexplored.

This paper addresses the ways that a bipartite graph G can be factored as a product $G \cong H \times K_2$ of a graph H with the complete graph K_2. This is in some ways analogous to factoring an even integer g into a product $g = h \cdot 2$, except that for graphs the factorization need not be unique. For example, Figure 1 shows that the 10-cycle $G = C_{10}$ can be prime factored as $G \cong H \times K_2$ where H can be either the path P_5 with loops at each end, or the cycle C_5. As we shall see, in general such graphs H arise in a simple way from the automorphism conjugacy classes of the involutions that reverse the bipartition of G. We will apply our results to the problem of extracting K_2 factors from even cycles and hypercubes.

We note that our current paper falls partly under the umbrella of [1]. Given arbitrary graphs H and K, with K bipartite, [1] classifies all the graphs H' for which $H \times K \cong H' \times K$. Replacing K with K_2 would seem to cover the topic of the current paper. However [1] employs a somewhat complex construction called the *factorial* of H, and thus in general the graphs H' appear to be difficult to compute. By contrast

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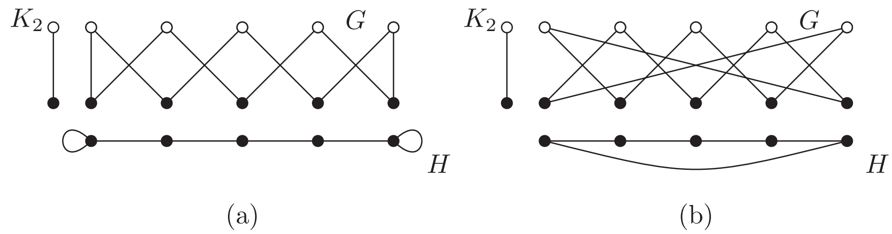

FIG. 1. *Two different prime factorizations of $G = C_{10}$.*

it turns out (as we shall see) that if $K = K_2$ the situation becomes much simpler, and the factorial is not required.

The reader is assumed to have some experience with direct products. See [3] for the definitive survey.

2. Notational preliminaries. The ideas in this paper require careful attention to how automorphisms of a bipartite graph act on the partite sets, and therefore we begin by laying out some notation that is specially catered to this.

Suppose G is a connected bipartite graph with partite sets X and Y. We regard the bipartition as an ordered pair (X, Y), so (Y, X) is a different bipartition. In discussing G we have in mind a definite bipartition (X, Y). Any automorphism $\alpha \in \mathrm{Aut}(G)$ must respect the bipartition in the sense that either $\alpha(X) = X$ and $\alpha(Y) = Y$, or $\alpha(X) = Y$ and $\alpha(Y) = X$. We call an automorphism of the first type a *preserving automorphism* and denote the collection of them as

$$\mathrm{PA}(G) = \{\alpha \in \mathrm{Aut}(G) : \alpha(X) = X, \alpha(Y) = Y\}.$$

Similarly, the set of *reversing automorphisms* is

$$\mathrm{RA}(G) = \{\alpha \in \mathrm{Aut}(G) : \alpha(X) = Y, \alpha(Y) = X\},$$

so $\mathrm{Aut}(G)$ is the disjoint union $\mathrm{PA}(G) \cup \mathrm{RA}(G)$. Of course $\mathrm{RA}(G)$ may be empty, but $\mathrm{PA}(G)$ at least contains the identity. An *involution* of G is an automorphism α of G such that $\alpha^2 = \mathrm{id}$. As above, we divide the set of involutions of G into the set of *preserving involutions* and the set of *reversing involutions*:

$$\mathrm{PI}(G) = \{\alpha \in \mathrm{PA}(G) : \alpha^2 = \mathrm{id}\},$$
$$\mathrm{RI}(G) = \{\alpha \in \mathrm{RA}(G) : \alpha^2 = \mathrm{id}\}.$$

(We remark in passing that reversing involutions play a key role in [2].) Given any $\alpha \in \mathrm{Aut}(G)$, we denote its restrictions to X and Y as α_X and α_Y, respectively. We agree to identify the codomains of α_X and α_Y with their images, so that both α_X and α_Y are bijective.

It will be convenient to regard any automorphism $\alpha \in \mathrm{Aut}(G)$ as an ordered pair $\alpha = (\alpha_X, \alpha_Y)$. Sometimes we will simply write $\alpha = (\alpha, \alpha)$ with the understanding that the first component is the restriction to X and the second is the restriction to Y. The following inversion formulas follow immediately:

$$\text{if } \alpha \in \mathrm{PA}(G), \text{ then } \alpha^{-1} = \left(\alpha_X^{-1}, \alpha_Y^{-1}\right),$$
$$\text{if } \alpha \in \mathrm{RA}(G), \text{ then } \alpha^{-1} = \left(\alpha_Y^{-1}, \alpha_X^{-1}\right).$$

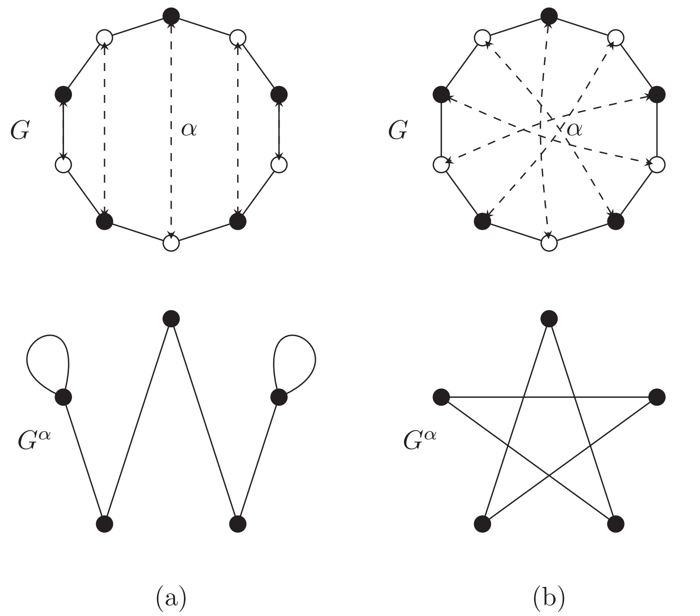

FIG. 2. *Examples of G and G^{α}.*

Given automorphisms α and β, their composition $\alpha\beta$ obeys the following rules:

$$\text{if } \beta \in \mathrm{PA}(G), \text{ then } \alpha\beta = (\alpha_X, \alpha_Y)(\beta_X, \beta_Y) = (\alpha_X\beta_X, \alpha_Y\beta_Y),$$
$$\text{if } \beta \in \mathrm{RA}(G), \text{ then } \alpha\beta = (\alpha_X, \alpha_Y)(\beta_X, \beta_Y) = (\alpha_Y\beta_X, \alpha_X\beta_Y).$$

We put $V(K_2) = \{0, 1\}$. For each $\epsilon \in V(K_2)$ we say $\overline{\epsilon} = 1-\epsilon$, so $\overline{0} = 1$ and $\overline{1} = 0$, and the map $\epsilon \mapsto \overline{\epsilon}$ is the reversing involution of K_2. Also, let us agree now that whenever we are discussing an edge xy of G, the vertex on the left (x) is assumed to be in X, and the vertex on the right (y) is assumed to be in Y. This convention is helpful when we write our automorphisms in pair form $\alpha = (\alpha, \alpha)$.

3. Extracting a K_2 factor. We now show how to factor a bipartite graph G as $H \times K_2$, provided such a factoring is possible. The following construction is a key ingredient. Given a reversing involution $\alpha \in \mathrm{RI}(G)$, we construct a graph G^{α} as follows:

$$V(G^{\alpha}) = X,$$
$$E(G^{\alpha}) = \{x\alpha(y) : xy \in E(G), x \in X, y \in Y\}.$$

Notice that G^{α} has half as many vertices as G. As an example, consider $G = C_{10}$ which is illustrated in Figures 2(a) and 2(b). The upper parts of each figure show G with vertices X colored black and vertices Y colored white. In each case a reversing involution α is indicated. The figures at the bottom show the corresponding graphs G^{α}.

We note in passing to the reader who may be bothered by the apparent arbitrary choice of X rather than Y as a vertex set for G^{α}, that we could equally well define a graph G_{α} with $V(G_{\alpha}) = Y$ and $E(G_{\alpha}) = \{\alpha(x)y : xy \in E(G), x \in X\ , y \in Y\}$. Then α restricts to an isomorphism $G^{\alpha} \to G_{\alpha}$. We will thus be content to work with G^{α} instead of G_{α}.

Our interest in the graphs G^α can be explained by comparing Figures 1 and 2. In Figure 2, the two graphs G^α are precisely the two factors H in Figure 1 which give $G \cong H \times K_2$. Indeed, the following proposition shows that this is a general property.

PROPOSITION 1. *If G is a connected bipartite graph, then $G \cong H \times K_2$ if and only if $H \cong G^\alpha$ for some reversing involution $\alpha \in \mathrm{RI}(G)$.*

Proof. Suppose $G \cong H \times K_2$. Then there is certainly no harm in assuming $G = H \times K_2$, and we do so. Thus $V(G) = V(H) \times \{0,1\}$ and $E(G) = \{(h,0)(h',1) : hh' \in E(H)\}$. The partite sets of G are $X = V(H) \times \{0\}$ and $Y = V(H) \times \{1\}$. Notice that the map $\alpha : (h,\epsilon) \mapsto (h,\overline{\epsilon})$ is a reversing involution in $\mathrm{RI}(G)$. By definition of G^α we have $V(G^\alpha) = V(H) \times \{0\}$ and $E(G^\alpha) = \{(h,0)(h',0) : hh' \in E(H)\}$. Clearly the map $\varphi : V(G^\alpha) \to V(H)$ defined as $\varphi(h,0) = h$ is an isomorphism from G^α to H.

Conversely, suppose $\alpha \in \mathrm{RI}(G)$. We just need to show $G \cong G^\alpha \times K_2$. As usual, we denote the bipartition of G as (X,Y). Define a map $\varphi : V(G) \to X \times \{0,1\}$ as $\varphi = ((\mathrm{id},0),(\alpha,1))$. (That is, φ sends any $x \in X$ to $(x,0)$ and any $y \in Y$ to $(\alpha(y),1)$.) This is bijective, and moreover

$$\begin{aligned} xy \in E(G) &\iff x\alpha(y) \in E(G^\alpha) \\ &\iff (x,0)(\alpha(y),1) \in E(G^\alpha \times K_2) \\ &\iff \varphi(x)\varphi(y) \in E(G^\alpha \times K_2), \end{aligned}$$

so $\varphi : G \to G^\alpha \times K_2$ is an isomorphism. □

So we now know that the graphs H for which $G \cong H \times K_2$ are precisely $H \cong G^\alpha$ for $\alpha \in \mathrm{RI}(G)$. But the correspondence between such graphs H and the elements of $\mathrm{RI}(G)$ is not necessarily bijective. It is quite possible that we could have $G^\alpha \cong G^\beta$ for distinct α and β. Our next proposition explains how such α and β are related. In reading the proof the reader is advised to keep in mind that $\beta^2 = \mathrm{id}$ implies that $xx' \in E(G^\beta)$ if and only if $\beta(x)x' \in E(G)$ and $x\beta(x') \in E(G)$.

PROPOSITION 2. *If $\alpha, \beta \in \mathrm{RI}(G)$, then $G^\alpha \cong G^\beta$ if and only if $\sigma\alpha\sigma^{-1} = \beta$ for some $\sigma \in \mathrm{Aut}(G)$.*

Proof. Suppose $G^\alpha \cong G^\beta$. Take an isomorphism $\widetilde{\sigma} : G^\alpha \to G^\beta$ so, in particular, $\widetilde{\sigma}$ is a bijection from X to X. Define $\sigma : V(G) \to V(G)$ as $\sigma = (\beta\widetilde{\sigma},\ \widetilde{\sigma}\alpha)$. It is easy to check that σ is bijective, and it is an automorphism of G as follows. (Recall that by convention any edge xy of G is assumed to satisfy $x \in X$ and $y \in Y$.)

$$\begin{aligned} xy \in E(G) &\iff x\alpha(y) \in E(G^\alpha) \\ &\iff \widetilde{\sigma}(x)\widetilde{\sigma}\alpha(y) \in E(G^\beta) \\ &\iff \beta\widetilde{\sigma}(x)\widetilde{\sigma}\alpha(y) \in E(G) \\ &\iff \sigma(x)\sigma(y) \in E(G). \end{aligned}$$

Therefore $\sigma \in \mathrm{Aut}(G)$. In fact $\sigma = (\beta\widetilde{\sigma},\ \widetilde{\sigma}\alpha) \in \mathrm{RA}(G)$ because $\alpha, \beta \in \mathrm{RA}(G)$. To complete the first part of the proof, observe that

$$\begin{aligned} \sigma\alpha\sigma^{-1} &= (\beta\widetilde{\sigma},\ \widetilde{\sigma}\alpha)(\alpha,\alpha)\left(\alpha\widetilde{\sigma}^{-1}, \widetilde{\sigma}^{-1}\beta\right) \\ &= (\beta\widetilde{\sigma},\ \widetilde{\sigma}\alpha)\left(\alpha\alpha\widetilde{\sigma}^{-1}, \alpha\widetilde{\sigma}^{-1}\beta\right) \\ &= \left(\beta\widetilde{\sigma}\alpha\alpha\widetilde{\sigma}^{-1},\ \widetilde{\sigma}\alpha\alpha\widetilde{\sigma}^{-1}\beta\right) = (\beta,\beta) = \beta. \end{aligned}$$

For the converse assume that $\alpha, \beta \in \mathrm{RI}(G)$ and $\sigma\alpha\sigma^{-1} = \beta$ for some $\sigma \in \mathrm{Aut}(G)$. Then $\sigma\alpha = \beta\sigma$. We consider two cases according to whether σ is in $\mathrm{PA}(G)$ or $\mathrm{RA}(G)$.

Case 1. Suppose $\sigma \in \mathrm{PA}(G)$. Then $xx' \in E(G^\alpha) \iff x\alpha(x') \in E(G) \iff \sigma(x)\sigma\alpha(x') \in E(G) \iff \sigma(x)\beta\sigma(x') \in E(G) \iff \sigma(x)\sigma(x') \in E(G^\beta)$. Thus σ restricts to an isomorphism from G^α to G^β.

Case 2. Suppose $\sigma \in \mathrm{RA}(G)$. Then $xx' \in E(G^\alpha) \iff x\alpha(x') \in E(G) \iff \sigma(x)\sigma\alpha(x') \in E(G) \iff \sigma(x)\beta\sigma(x') \in E(G) \iff \beta\sigma(x)\sigma(x') \in E(G) \iff \beta\sigma(x)\beta\sigma(x') \in E(G^\beta)$. Thus $\beta\sigma$ restricts to an isomorphism from G^α to G^β. ∎

The previous two propositions tell us how to compute, up to isomorphism, all graphs H for which $G = H \times K_2$: Compute $\mathrm{RI}(G)$ and determine its orbits under the action of $\mathrm{Aut}(G)$ by conjugation. Then take a representative α from each orbit and form the graphs $H = G^\alpha$. We summarize this as a theorem.

THEOREM 1. *Suppose G is a connected bipartite graph. The set of all graphs H for which $G \cong H \times K_2$ can be found with the following process.*

1. *Compute* $\mathrm{RI}(G)$. *If* $\mathrm{RI}(G) = \emptyset$, *then K_2 cannot be factored from G. Otherwise proceed as follows.*
2. *The group* $\mathrm{Aut}(G)$ *acts on* $\mathrm{RI}(G)$ *by conjugation. Take representatives $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the orbits of this action.*
3. *The graphs H for which $G \cong H \times K_2$ are precisely $H = G^{\alpha_1}, G^{\alpha_2}, \ldots, G^{\alpha_n}$.*

We now illustrate this theorem by carrying out the process with even cycles and hypercubes.

4. First example: Factoring even cycles. As an application of these ideas, we examine the problem of factoring cycles C_n where $n = 2q$. Imagine C_n as a regular n-gon centered at the origin of the plane. We can identify $\mathrm{Aut}(C_n)$ with the dihedral group D_{2n}. In what follows we will need to know what the conjugacy classes of D_{2n} look like; thus we begin with a little notation. First label the vertices of the n-gon consecutively as $1, 2, 3, \ldots, n$ in the clockwise direction. Let μ denote the reflection about the line passing through the center of the n-gon and the vertex 1, and let ρ be the clockwise rotation of $\frac{2\pi}{n}$ radians. Clearly each element of D_{2n} can be expressed as a product $\mu^i\rho^j$ for some $0 \le i \le 1$ and $0 \le j \le 2q-1$. Since n is even we note that the center of D_{2n} consists of two elements, the identity and the rotation ρ^q; thus they are each in conjugacy classes by themselves. A few simple calculations show that there are two conjugacy classes of reflections, $\{\mu\rho^{2j} : 1 \le j \le q\}$ and $\{\mu\rho^{2j-1} : 1 \le j \le q\}$, and q conjugacy classes of rotations of the form $\{\rho^k, \rho^{-k}\}$ for $1 \le k \le q$.

To apply our results, we need to investigate the involutions of C_n. Any non-identity involution of C_n is either a reflection or the rotation ρ^q by π radians. Half of the reflections (the ones that fix two vertices) are in $\mathrm{PI}(C_n)$, and the other half (the ones that fix no vertices) are in $\mathrm{RI}(C_n)$. The involution ρ^q is in $\mathrm{RI}(C_n)$ if and only if q is odd. Therefore the structure of $\mathrm{RI}(C_n)$ depends on the parity of q. We consider each case separately.

If q is even, $\mathrm{RI}(C_n)$ consists of q reflections. Since all of these reflections are conjugate to each other, C_n factors uniquely as $C_n \cong C_n^\alpha \times K_2$, where α is any element of $\mathrm{RI}(C_n)$. One quickly checks that (as in Figure 2(a)) C_n^α is the path of length q with loops at each end. This graph is easily seen to be prime, so we have a prime factorization $C_n \cong H \times K_2$, where H is a length-q path with loops at each end. This is the only way C_n can be factored with a K_2 as a factor.

If q is odd, then $\mathrm{RI}(C_n)$ consists of q reflections plus the rotation ρ^q. We know that the reflections are all conjugate to each other and that the rotation lies in a separate conjugacy class. Thus C_n can factor in exactly two ways, either as $C_n \cong C_n^\alpha \times K_2$,

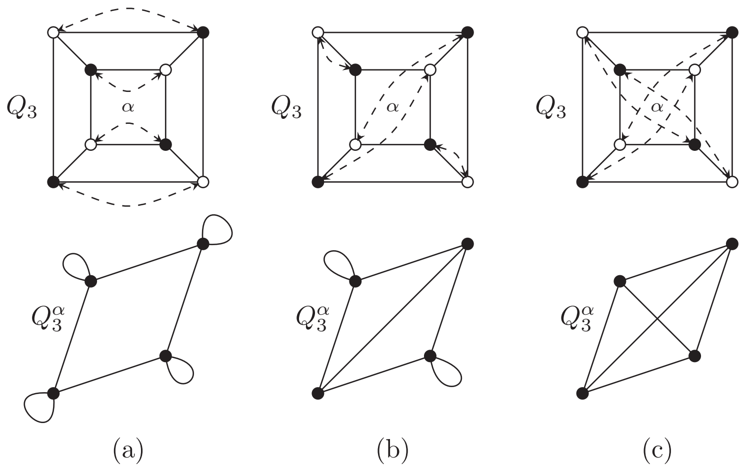

FIG. 3. *Examples of* Q_3 *and* Q_3^α.

where α is a reflection, or as $C_n \cong C_n^{\rho^q} \times K_2$. As before, C_n^α is a length-q path with loops at each end. It is easy to verify that $C_n^{\rho^q} \cong C_q$, giving $C_n \cong C_q \times K_2$.

5. Second example: Factoring hypercubes. The *n-dimensional cube* Q_n is the graph for which $V(Q_n)$ is the set of all 0-1 sequences of length n, with two vertices being adjacent if and only if they differ in exactly one position. We can regard Q_n as the Cartesian product of n copies of K_2. This graph is bipartite, with one partite set X being the set of all vertices with an even number of 1's and the other partite set Y being the set of all vertices with an odd number of 1's.

We now investigate the problem of factoring Q_n as $Q_n = H \times K_2$. This has already been addressed in [4, 5] using a different line of reasoning. In these papers it was shown that all decompositions of Q_n into a direct product are of the form $H \times K_2$ and that H is a hypercube of dimension $n-1$ with certain additional edges defined by involutions. However the question of direct computation and enumeration of all such graphs H was left open. Since we are merely refining known results, the exposition in this section will be somewhat informal.

For a starting point consider Q_3. Theorem 1 asserts that $Q_3 = H \times K_2$ exactly when $H = Q_3^\alpha$ for some reversing involution α of Q_3. Figure 3 shows three such α along with the corresponding graphs $H = Q_3^\alpha$. (The reader can check that $H \times K_2 \cong Q_3$ in each case.) Thus there are at least three H for which $Q_3 = H \times K_2$, but we are immediately confronted with the questions of whether these are the only three, and how this would play out for an arbitrary Q_n.

To apply the full force of Theorem 1 to Q_n, we must investigate the orbits of the conjugation action of $\mathrm{Aut}(Q_n)$ on $\mathrm{RI}(Q_n)$. This in turn involves understanding the structure of maps in both $\mathrm{Aut}(Q_n)$ and $\mathrm{RI}(Q_n)$. We now focus on this task.

We denote an arbitrary vertex of Q_n by a sequence $x_1x_2\dots x_n$, where each x_i is either 0 or 1. By Theorem 4.15 of [3] any $f \in \mathrm{Aut}(Q_n)$ has expression

$$f(x_1x_2\dots x_n) = \delta_1(x_{\pi(1)})\delta_2(x_{\pi(2)})\dots\delta_3(x_{\pi(n)}), \tag{5.1}$$

where π is a permutation of $\{1, 2, \dots, n\}$ and for each i the function $\delta_i \in \mathrm{Aut}(K_2)$ is either the identity $\delta_i(\varepsilon) = \varepsilon$ or the nontrivial automorphism $\delta_i(\varepsilon) = \overline{\varepsilon}$. Intuitively, this means that the automorphism group of the Cartesian product of n K_2's is generated

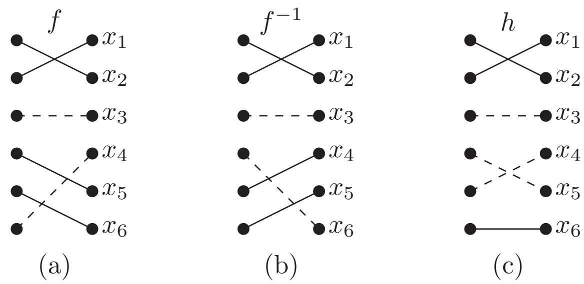

FIG. 4. *Graphic representation of automorphisms of Q_6.*

by the automorphisms δ_i of the individual factors together with the permutations of the factors. (It is, in fact, isomorphic to the automorphism group of the disjoint union of n K_2's.)

As an example, consider the automorphism of Q_6 defined as $f(x_1x_2x_3x_4x_5x_6) = x_2x_1\overline{x_3}x_5x_6\overline{x_4}$. We can represent f graphically as in Figure 4(a). The solid line from x_1 indicates that f places the value of x_1 in the second position of the output list, and the solid line from x_2 indicates that f places the value of x_2 in the first position of the output list. The dashed line from x_3 indicates that f places the value $\overline{x_3}$ in the third position, and so on. In general, a dashed line extending leftward from x_i to the jth position means f places the value $\overline{x_i}$ in the jth position of the output list. A solid line extending leftward from x_i to the jth position means f places the value x_i in the jth position of the output list. (The fact that the x_i's appear on the right, rather than the left, is a consequence of the unfortunate convention of writing functions in postfix form $f(x)$ rather than in prefix $(x)f$.)

The graphic representation of f^{-1} is the reflection of the graphic representation of f across the vertical axis. (Figures 4(a) and 4(b) show the graphic representation of an f and f^{-1}.) Thus f is an involution if and only if $f = f^{-1}$ if and only if the graphic representation of f is symmetric with respect to the vertical axis. (Figure 4(c) shows a typical involution.)

From this it follows that if f is an involution, then the permutation π from (5.1) is a product of disjoint transpositions. Moreover, if π interchanges i and j, then the graphic representation of f has lines extending from the ith and jth positions on the right to the jth and ith positions on the left, *and* these two lines are either both solid or both dashed.

We now identify certain "canonical" involutions of Q_n. Given positive integers j and k with $0 \leq j + 2k \leq n$, let $\alpha_{j,k} \in \mathrm{Aut}(Q_n)$ have a graphic representation consisting of j horizontal dashed lines followed by k consecutive involutions with solid lines, followed by $n - j - 2k$ horizontal solid lines. Several such involutions of Q_9 are illustrated in Figure 5(a)–(c).

Figure 5(d) illustrates how for an involution f of Q_n, one can find some $\sigma \in \mathrm{Aut}(Q_n)$ for which $\sigma^{-1}f\sigma = \alpha_{j,k}$. Thus every involution is conjugate to some $\alpha_{j,k}$. On the other hand no two distinct $\alpha_{j,k}$ are conjugate to each other: Each $\alpha_{j,k}$ is uniquely determined by integers j and k, and it's straightforward to check $\sigma^{-1}\alpha_{j,k}\sigma(x_1x_2 \dots x_n)$ fixes exactly $n - j - 2k$ of the positions in the list $x_1x_2 \dots x_n$ and changes j of the positions x_i to $\overline{x_i}$. Therefore if $\sigma^{-1}\alpha_{j,k}\sigma = \alpha_{j',k'}$, then $j' = j$ and $k' = k$.

Which of the $\alpha_{j,k}$ are reversing involutions? Since one partite set of Q_n consists of vertices $x_1x_2 \dots x_n$ that have an even number of 1's and the other partite set has

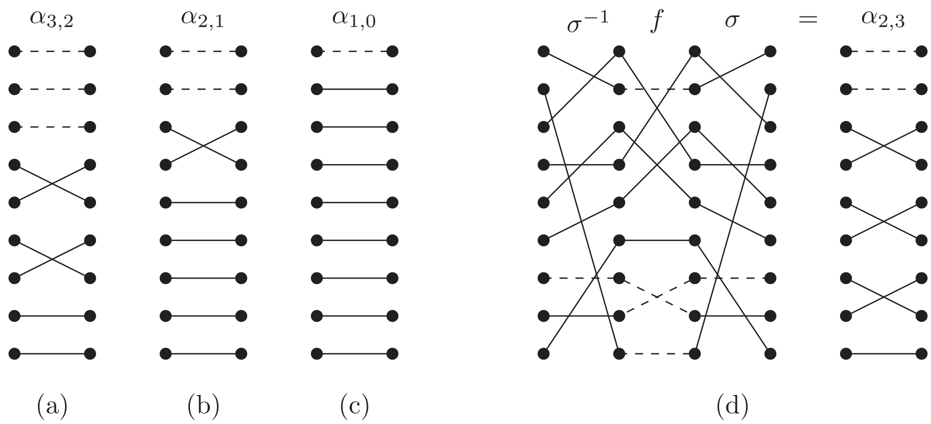

FIG. 5. *Canonical involutions* $\alpha_{j,k}$.

all the vertices with an odd number of 1's, the only $\alpha_{j,k}$ that reverse the parity are those for which j is odd.

The previous two paragraphs imply that each $\mathrm{Aut}(Q_n)$-conjugacy class of $\mathrm{RI}(Q_n)$ contains exactly one $\alpha_{j,k}$, and j is odd; conversely each such $\alpha_{j,k}$ belongs to a conjugacy class of $\mathrm{RI}(Q_n)$.

It follows that the number of conjugacy classes of $\mathrm{RI}(Q_n)$ equals the number of automorphisms $\alpha_{j,k}$ of Q_n with j odd. These are easy to count, for their number equals the number of non-negative integer solutions to $1 \le j + 2k \le n$ with j odd. Replacing the odd j with $2m + 1$, we are looking for the number of non-negative solutions to $1 \le 2m + 1 + 2k \le n$, or rather to $0 \le m + k \le \frac{n-1}{2}$. The number of solutions equals the number of integer lattice points in the first quadrant of the m-k plane which lie on or below the line $m + k = \frac{n-1}{2}$. There are $\frac{1}{2}\lceil\frac{n}{2}\rceil(\lceil\frac{n}{2}\rceil + 1)$ such points. Applying this to Theorem 1 gives a result that describes all the ways that a K_2 factor can be extracted from a hypercube.

THEOREM 2. *The graphs H for which $Q_n = H \times K_2$ are precisely the graphs $H = Q_n^{\alpha_{j,k}}$ for odd j. Up to isomorphism, there are exactly $\frac{1}{2}\lceil\frac{n}{2}\rceil(\lceil\frac{n}{2}\rceil + 1)$ such graphs H.*

For example, if $n = 3$, then there are three such H, so Figure 3 indeed illustrates all the $H = Q_3^{\alpha}$ for which $Q_3 \cong H \times K_2$.

Although it would not be difficult to compute the graphs $Q_n^{\alpha_{j,k}}$ from scratch, we offer a final result which underscores their surprisingly simple structure. Given any $\alpha_{j,k} \in \mathrm{Aut}(Q_n)$ with j odd, there is an associated automorphism $\alpha_{j-1,k} \in \mathrm{Aut}(Q_{n-1})$. Graphically, we obtain $\alpha_{j-1,k}$ by "throwing away" the top horizontal dashed line in $\alpha_{j,k}$. Since $j - 1$ is even we have $\alpha_{j-1,k} \in \mathrm{PA}(Q_{n-1})$. Our next proposition shows that $Q_n^{\alpha_{j,k}}$ is simply Q_{n-1} with the edges $\{x\alpha_{j-1,k}(x) : x \in V(Q_{n-1})\}$ appended to it.

PROPOSITION 3. *If j is odd, then $Q_n^{\alpha_{j,k}} \cong Q_{n-1} \cup \{x\alpha_{j-1,k}(x) : x \in V(Q_{n-1})\}$.*

Proof. We construct an isomorphism $\mu : Q_{n-1} \cup \{x\alpha_{j-1,k}(x) : x \in V(Q_{n-1})\} \to Q_n^{\alpha_{j,k}}$.

Let $\lambda_0 : V(Q_{n-1}) \to V(Q_n)$ be the map $\lambda_0(x) = 0x$ that appends a zero to the left of list x. Thus $\lambda_0(01101) = 001101$, etc. Similarly let $\lambda_1 : V(Q_{n-1}) \to V(Q_n)$ append a 1 to the left of its argument. Notice $\alpha_{j,k}\lambda_0 = \lambda_1\alpha_{j-1,k}$.

Let X be the partite set of Q_{n-1} consisting of 0-1 lists with an even number of 1's and let Y be the partite set consisting of all the lists with an odd number of 1's. Now define a bijection $\mu : Q_{n-1} \to Q_n^{\alpha_{j,k}}$ as $\mu = (\lambda_0,\ \alpha_{j,k}\lambda_0)$. (Recall the notation from section 2.) Notice that μ is a homomorphism, for given $xy \in E(Q_{n-1})$ (with $x \in X$) $\lambda_0(x)\lambda_0(y) \in E(Q_n)$, so $\mu(x)\mu(y) = \lambda_0(x)\alpha_{j,k}\lambda_0(y)$ is an edge of $Q_n^{\alpha_{j,k}}$ by definition of $Q_n^{\alpha_{j,k}}$.

But also note that μ carries any pair $x, \alpha_{j-1,k}(x)$ to the endpoints of an edge in $Q_n^{\alpha_{j,k}}$. If $x \in X$, then $\alpha_{j-1,k}(x) \in X$ also, and $\mu(x)\mu(\alpha_{j-1,k}(x)) = \lambda_0(x)\lambda_0(\alpha_{j-1,k}(x))$ $= \lambda_0(x)\alpha_{j,k}\lambda_1(x)$, which is an edge of $Q_n^{\alpha_{j,k}}$ because $\lambda_0(x)\lambda_1(x) \in E(Q_n)$. Similarly if $x \in Y$ we have $\mu(x)\mu(\alpha_{j-1,k}(x)) = \alpha_{j,k}\lambda_0(x)\alpha_{j,k}\lambda_0\alpha_{j-1,k}(x) = \alpha_{j,k}\lambda_0(x)\alpha_{j,k}\alpha_{j,k}\lambda_1(x)$ $= \alpha_{j,k}\lambda_0(x)\lambda_1(x) = \lambda_1(x)\alpha_{j,k}\lambda_0(x) \in E(Q_n^{\alpha_{j,k}})$.

So far we have seen that μ is an injective homomorphism from the graph $Q_{n-1} \cup \{x\alpha_{j-1,k}(x) : x \in V(Q_{n-1}\}$ to $Q_n^{\alpha_{j,k}}$. We now need only confirm that it is surjective. Notice that edges in $Q_n^{\alpha_{j,k}}$ fall into four categories. First, they may have form $0x\,\alpha_{j,k}(0y)$, where $x \in V(Q_{n-1})$ has an even number of 1's, y has an odd number of 1's, and $xy \in E(Q_{n-1})$. Second, they may have form $1x\,\alpha_{j,k}(1y)$, where $x \in V(Q_{n-1})$ has an odd number of 1's, y has an even number of 1's, and $xy \in E(Q_{n-1})$. Third, they may have form $0x\,\alpha_{j,k}(1x)$, where $x \in V(Q_{n-1})$ has an even number of 1's. Fourth, they may have form $1x\,\alpha_{j,k}(0x)$, where $x \in V(Q_{n-1})$ has an odd number of 1's. The following calculations show that in each case these edges are images under μ of edges in $Q_{n-1} \cup \{x\alpha_{j-1,k}(x) : x \in V(Q_{n-1}\}$.

$$\begin{aligned}
0x\,\alpha_{j,k}(0y) &= \lambda_0(x)\,\alpha_{j,k}\lambda_0(y) = \mu(x)\mu(y),\\
1x\,\alpha_{j,k}(1y) &= \alpha_{j,k}^2\lambda_1(x)\,\alpha_{j,k}\lambda_1(y) = \alpha_{j,k}\lambda_0\alpha_{j-1,k}(x)\,\lambda_0\alpha_{j-1,k}(y)\\
&= \mu\big(\alpha_{j-1,k}(x)\big)\mu\big(\alpha_{j-1,k}(y)\big),\\
0x\,\alpha_{j,k}(1x) &= \lambda_0(x)\,\alpha_{j,k}\lambda_1(x) = \lambda_0(x)\,\lambda_0\alpha_{j-1,k}(x) = \mu(x)\mu\big(\alpha_{j-1,k}(x)\big),\\
1x\,\alpha_{j,k}(0x) &= \alpha_{j,k}^2\lambda_1(x)\,\alpha_{j,k}\lambda_0(x) = \alpha_{j,k}\lambda_0\alpha_{j-1,k}(x)\,\alpha_{j,k}\lambda_0(x)\\
&= \mu\big(\alpha_{j-1,k}(x)\big)\mu(x).
\end{aligned}$$

This completes the proof. ☐

Theorem 2 and Proposition 3 give a simple way to construct all graphs H for which $Q_n = H \times K_2$. Take any $\alpha_{j,k} \in \mathrm{Aut}(Q_{n-1})$ with j *even*, and let $H = Q_{n-1} \cup \{x\,\alpha_{j,k}(x) : x \in V(Q_{n-1})\}$. As an example Figure 6 shows the six graphs H for which $Q_5 = H \times K_2$. In each case the edges of Q_{5-1} are drawn bold, and the edges of form $x\alpha_{j,k}(x)$ are drawn lighter.

In [4, 5], an involution of Q_{n-1} is called *bipartite* if it preserves the bipartition of Q_{n-1}. It is proved there that $Q_n \cong H \times K_2$ if and only if there is a bipartite involution $\alpha \in \mathrm{Aut}(Q_{n-1})$ for which $H = Q_{n-1} \cup \{x\alpha(x) : x \in V(Q_{n-1})\}$. This agrees with our development here, but we have also completely enumerated and described the bipartite involutions α. We remark that [4, 5] also prove that $Q_n \cong H \times K_2$ is always a *prime* factorization of Q_n.

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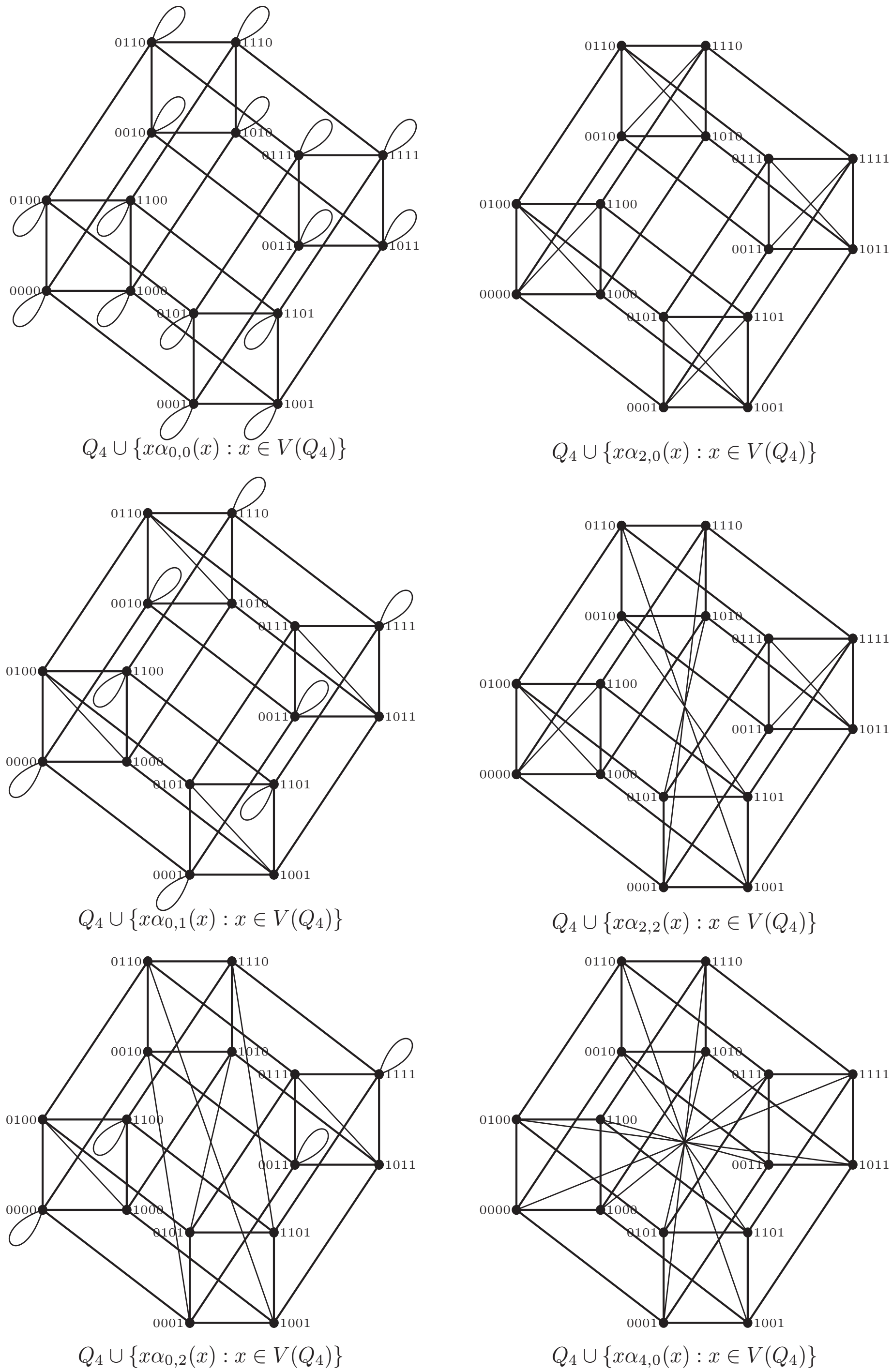

FIG. 6. *The six graphs H for which $Q_5 \cong H \times K_2$.*

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