On uniqueness of prime bipartite factors of graphs

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Abstract

It has long been known that the class of connected nonbipartite graphs (with loops allowed) obeys unique prime factorization over the direct product of graphs. Moreover, it is known that prime factorization is not necessarily unique in the class of connected bipartite graphs.

But any prime factorization of a connected bipartite graph has exactly one bipartite factor. Moreover, empirical evidence suggests that any two prime factorings of a given connected bipartite graph have isomorphic bipartite factors. This prompts us to conjecture that among all the different prime factorings of a given connected bipartite graph, the bipartite factor is always the same.

The present paper proves that the conjecture is true for graphs that have a K_2 factor. (Even in this simple case, the result is surprisingly nontrivial.) Further, we indicate how to compute all possible prime factorings of such a graph. In addition, we show how the truth of the conjecture (in general) would lead to a method of finding all distinct prime factorings of any connected bipartite graph.

To accomplish this, we prove the following preliminary result, which is the main technical result of the paper: Suppose $A \times B$ is connected and bipartite, and B is the bipartite factor. If $A \times B$ admits an involution that reverses partite sets, then B also admits such an involution.

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1. Introduction

We assume our reader is familiar with graph products, but to fix notation and terminology we review the main definitions here. For a survey, see [3].

Let Γ be the set of (isomorphism classes of) graphs without loops; thus $\Gamma \subset \Gamma_0$, where Γ_0 is the set of graphs with loops allowed. The *direct product* of graphs $A, B \in \Gamma_0$ is the graph $A \times B$ with vertices $V(A) \times V(B)$ and edges

$$E(A \times B) = \{(a, b)(a', b') \mid aa' \in E(A) \text{ and } bb' \in E(B)\}.$$

Figure 1 shows a typical example. The direct product is commutative and associative in the sense that the maps $(a, b) \mapsto (b, a)$ and $(a, (b, c)) \mapsto ((a, b), c)$ are isomorphisms $A \times B \cong B \times A$ and $A \times (B \times C) \cong (A \times B) \times C$, respectively.

Further, if K_1^* denotes a vertex on which there is a loop, then $K_1^* \times A \cong A$ for any graph A, so K_1^* is the unit for the direct product. A nontrivial graph $G \in \Gamma_0$ is *prime over* $\times$ if for any factoring $G = A \times B$ into graphs $A, B \in \Gamma_0$ it follows that one of A or B is K_1^* and the other is G.

A consequence of a fundamental result by McKenzie [7] is that every connected nonbipartite graph in Γ_0 factors over $\times$ uniquely into primes. Specifically, if $G = A_1 \times A_2 \times \cdots \times A_k$ and $G = B_1 \times B_2 \times \cdots \times B_\ell$ are two prime factorings of a connected nonbipartite graph G, then $k = \ell$ and $B_i \cong A_{\pi(i)}$ for some permutation π of $\{1, 2, \ldots, k\}$. McKenzie's paper involves general relational structures; for purely graph-theoretical proofs of unique prime factorization, see Imrich [5], or [3] for a more recent proof.

But if G is bipartite, its prime factorization may not be unique. Figure 2 illustrates this. It shows a graph G with prime factorings $G = A \times B$ and $G = A' \times B$ with $A \ncong A'$. Figure 3 shows another example, one pointing to the fact that the number of prime factors may vary with the factorization.

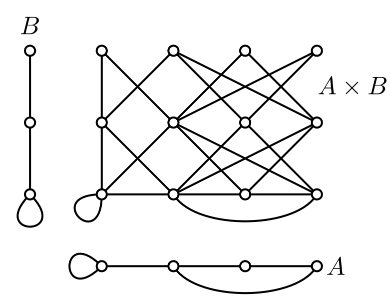


Figure 1: Direct product of graphs

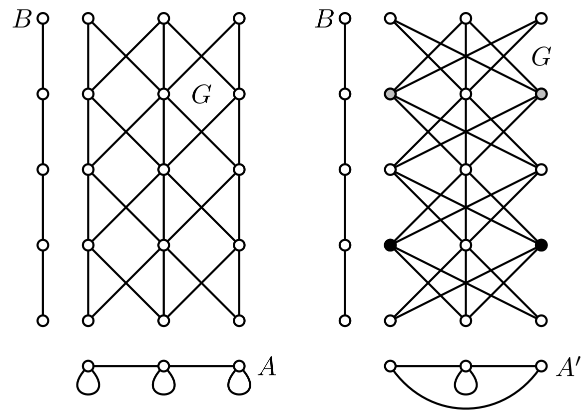


Figure 2: Two prime factorings $G = A \times B$ and $G = A' \times B$ of a bipartite graph G, with a common prime bipartite factor B. (The products are indeed isomorphic: The right can be transformed to the left by transposing the two black vertices and the two gray vertices.)

Let us examine factorings of bipartite graphs in more detail. An oft-used theorem by Weichsel states the following: Let A and B be connected graphs. Then $A \times B$ is connected if and only if at least one of A or B is not bipartite; if both A and B are bipartite, then $A \times B$ has exactly two components. Moreover, $A \times B$ is bipartite if and only if at least one factor is bipartite. (See Theorem 5.9 of [3] for a proof of Weichsel's theorem.)

It follows that if a connected bipartite graph G has a prime factoring $G = A_1 \times A_2 \times \cdots \times A_k$, then *exactly one* prime factor is bipartite. This is borne out in Figures 2 and 3. But notice that, in each example, although the prime factorings of G are different, the bipartite factor B is the same. These examples, plus additional evidence not presented here, prompt a conjecture.

Conjecture 1. *Given two prime factorings of a connected bipartite graph, the prime bipartite factors are isomorphic. In other words, the prime bipartite factor in a factoring of a connected bipartite graph is unique.*

The purpose of this paper is to prove that this conjecture is true for any graph that has a bipartite factor of K_2. That is, we will show that if a connected bipartite graph has a prime factoring that contains a K_2 factor, then any prime factoring of this graph has a K_2 factor. This result appears to be surprisingly nontrivial, but we hope that the proof presented here will suggest a proof of the conjecture in general.

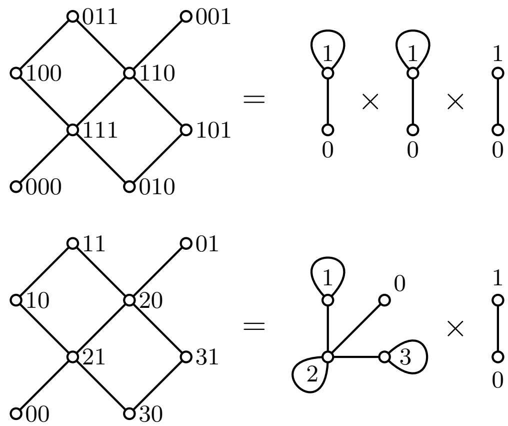


Figure 3: Two prime factorings with common bipartite factor is K_2.

In fact, we will prove slightly more than the uniqueness of a K_2 factor. Given any connected graph with a K_2 factor, we will indicate how to find *all* prime factorings of the graph.

2. Main Results

We now outline the proof of our main result: for any connected bipartite graph, if one prime factoring has bipartite factor K_2, then any prime factoring of the graph has bipartite factor K_2. Our proof will hinge on the following lemma. Its origins are unclear; it has been reproved many times.

Lemma 1. *A connected bipartite graph B has a factoring $B \cong H \times K_2$ in Γ_0 if and only if B admits an involution (i.e. an automorphism of order 2) that interchanges its partite sets.*

Proof. Let $V(K_2) = \{0, 1\}$. If $B \cong H \times K_2$, then there is an involution of $H \times K_2$ defined as $(x, 0) \mapsto (x, 1)$ and $(x, 1) \mapsto (x, 0)$, and this interchanges the partite sets $V(H) \times \{0\}$ and $V(H) \times \{1\}$.

Conversely, let there be an involution $\sigma : B \to B$ interchanging partite sets B_0 and B_1 of B. Then there is a partition $\Sigma = \{\{x, \sigma(x)\} \mid x \in V(B)\}$ of $V(B)$ consisting of two-element sets fixed by σ. Form the quotient B/Σ, with vertex set Σ and where $\{x, \sigma(x)\}$ is adjacent to $\{y, \sigma(y)\}$ precisely if some edge of B joins them. It is straightforward to check that the map $f : B \to B/\Sigma \times K_2$ defined as

$$f(x) = \begin{cases} (\{x, \sigma(x)\}, 0) & \text{if } x \in B_0, \\ (\{x, \sigma(x)\}, 1) & \text{if } x \in B_1 \end{cases}$$

is an isomorphism. ■

The following is the main technical result of this paper.

Theorem 1. *Suppose A and B are connected graphs, and B is bipartite but A is not. If $A \times B$ admits an involution that reverses its partite sets, then B also admits an involution that reverses its partite sets.*

To preserve continuity of exposition, we postpone the proof of Theorem 1. Instead, we use Theorem 1 now to quickly reach our main objective:

Theorem 2. *Suppose G is an arbitrary connected bipartite graph. If G has a K_2 factor, then the bipartite factor in any prime factoring of G is a K_2. In other words, if $G \cong A \times B \cong C \times K_2$, and B is prime and bipartite, then $B \cong K_2$.*

Proof. If $G \cong A \times B \cong C \times K_2$, then $A \times B$ has an involution that reverses its bipartition because $C \times K_2$ does. By Theorem 1, B also admits an involution that reverses its partite sets. Then $B \cong H \times K_2$, by Lemma 1. As B is prime, it follows that $H \cong K_1^*$, so $B \cong K_2$. ■

It remains to prove Theorem 1. The theorem may sound quite plausible, but its proof is not a trivial matter. The next several sections recall some machinery that will be needed in the proof. Section 3 reviews the idea of the Cartesian product of graphs, and of prime factoring over this product. Section 4 then discusses a certain equivalence relation R on the vertex set of a graph. This leads to the notion of the Cartesian skeleton of a graph, in Section 5. All of this is put to use in Section 6, where Theorem 1 is proved, in essence by reducing factorization over the direct product to factorization over the Cartesian product.

The concluding Section 7 explains how to find all prime factorings of a given connected bipartite graph.

3. The Cartesian Product

The proof of Theorem 1 will employ the Cartesian product of graphs. The *Cartesian product* of graphs $A, B \in \Gamma$ is the graph $A \Box B$ with vertices $V(A) \times V(B)$ and edges

$$E(A \Box B) = \{(a,b)(a',b') \mid aa' \in E(A) \text{ and } b = b', \text{ or } a = a' \text{ and } bb' \in E(B)\}.$$

(See Figure 4.) The Cartesian product is commutative and associative in the sense that $A \Box B \cong B \Box A$ and $A \Box (B \Box C) \cong (A \Box B) \Box C$. Letting $B + C$ denote the disjoint union of graphs B and C, we also get the distributive law

$$A \Box (B + C) = A \Box B + A \Box C. \tag{1}$$

Observe that this is true equality, rather than mere isomorphism.

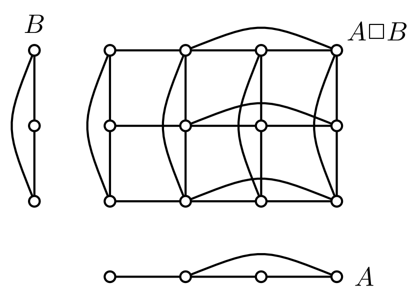


Figure 4: A Cartesian product of graphs.

Clearly $K_1 \Box A \cong A$ for any graph A, so K_1 is the unit for the Cartesian product. A nontrivial graph G is *prime over* $\Box$ if for any factoring $G = A \Box B$, one of A or B is K_1. Certainly every graph can be factored into (possibly more than two) prime factors in Γ. Sabidussi and Vizing [8, 9] proved that each connected graph has a unique prime factoring (in Γ), up to order and isomorphism of the factors. More precisely, we have the following.

Theorem 3 (Theorem 6.8 of [3]). *Let $G, H \in \Gamma$ be isomorphic connected graphs with prime factorings $G = G_1 \Box \cdots \Box G_k$ and $H = H_1 \Box \cdots \Box H_\ell$. Then $k = \ell$, and for any isomorphism $\varphi : G \to H$, there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to H_i$ for which*

$$\varphi(x_1, x_2, \ldots, x_k) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})).$$

We emphasize that discussions of prime factorizations over $\Box$ concern only graphs in Γ. Indeed, the above theorem is false in the class Γ_0. To see this, let K_2^* be K_2 with loops at each vertex, and note that $K_2 \Box K_2^* \cong K_2^* \Box K_2^*$ are distinct prime factorings of the square with loops at each vertex.

Theorem 3 invites us to identify each H_i with $G_{\pi^{-1}(i)}$, yielding a corollary.

Corollary 1. *If $\varphi : G_1 \Box \cdots \Box G_k \to H_1 \Box \cdots \Box H_k$ is an isomorphism, and each G_i and H_i is prime, then the vertices of each H_i can be relabeled so that*

$$\varphi(x_1, x_2, \ldots, x_k) = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$$

for some permutation π of $\{1, \ldots, k\}$.

4. *R*-Thin Graphs

The notion of so-called R-thinness is an important issue in factorings over the direct product. McKenzie [7] used this idea (in a somewhat more general form), citing an earlier use by Chang [1]. To motivate this topic, notice that in Theorem 3 any component function φ_i depends on only one variable x_j. This places stringent (and useful) restrictions on the isomorphism φ. The corresponding result for the direct product is generally false without the mild restriction of R-thinness (defined below). Indeed, in Figure 5 (left), the transposition of vertices $a2$ and $c2$ is an isomorphism $A \times B \to A \times B$ that is not of the form prescribed by Theorem 3.

This transposition is possible because $a2$ and $c2$ have the same neighborhood. Evidently, then, vertices with identical neighborhoods complicate the discussion of prime factorizations over the direct product. To overcome this difficulty, one forms a relation R on the vertices of a graph. Two vertices x

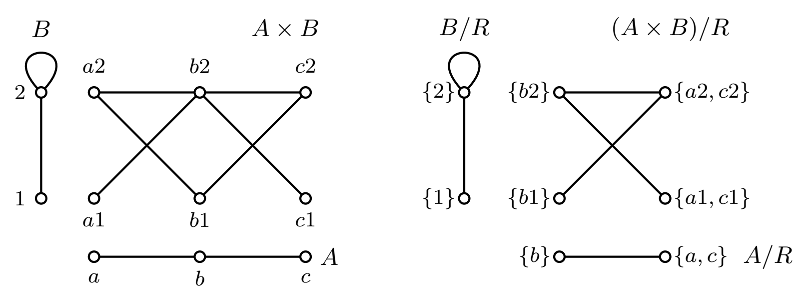


Figure 5: Graphs A, B and $A \times C$ (left), and quotients $A/R, B/R$ and $(A \times B)/R$ (right).

and x' of a graph G are in *relation* R, written xRx', precisely if their open neighborhoods are identical, that is, if $N_G(x) = N_G(x')$. It is a simple matter to check that R is an equivalence relation on $V(G)$. (For example, the equivalence classes for $A \times B$ in Figure 5 are $\{b1\}$, $\{b2\}$, $\{a1, c1\}$ and $\{a2, c2\}$; those of A are $\{a, c\}$ and $\{b\}$, and those of B are $\{1\}$ and $\{2\}$.)

Given a graph G, we define a quotient graph G/R (in Γ_0) whose vertex set is the set of R-equivalence classes of G, and for which two classes are adjacent if they are joined by an edge of G. (And a single class carries a loop provided that an edge of G has both endpoints in that class.) Figure 5 shows quotients A/R, B/R and $(A \times B)/R$. A graph G is called *R-thin* if all of its R-equivalence classes contain just one vertex. In this case $G/R \cong G$. In Figure 5, B is R-thin. It is easily verified that G/R is R-thin for any $G \in \Gamma_0$.

We note in passing that an analogue of Theorem 3 for the direct product holds for R-thin nonbipartite graphs. See Theorem 8.15 of [3]. (More generally, McKenzie [7] proves that thin structures in a class $\mathbf{Q}$, which contains the nonbipartite connected graphs, have a "strict refinement property." The analogue of our Theorem 1 for nonbipartite graphs, and indeed the abovementioned Theorem 8.15 of [3], can be seen as consequences of this.)

Given $x \in V(G)$ let $[x] = \{x' \in V(G) \mid N_G(x') = N_G(x)\}$ denote the R-equivalence class containing x. As the relation R is defined entirely in terms of adjacencies, it is clear that given an isomorphism $\varphi : G \to H$ we have xRy in G if and only if $\varphi(x) R \varphi(y)$ in H. Thus φ maps R-equivalence classes of G to R-equivalence classes of H, and in particular $\varphi([x]) = [\varphi(x)]$. Thus any isomorphism $\varphi : G \to H$ induces an isomorphism $\widetilde{\varphi} : G/R \to H/R$ defined as $\widetilde{\varphi}([x]) = [\varphi(x)]$. But an isomorphism $\widetilde{\varphi} : G/R \to H/R$ does not necessarily imply that there is an isomorphism $\varphi : G \to H$. (Consider $G = P_3$ and $H = K_2$.) However, we do have the following result in this direction. The straightforward proof can be found in Section 8.2 of [3].

Proposition 1. *Given an isomorphism $\widetilde{\varphi} : G/R \to H/R$, with $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/R)$, there is also an isomorphism $\varphi : G \to H$. (Any such φ can be obtained from $\widetilde{\varphi}$ by declaring that φ restricts to a bijection $X \to \widetilde{\varphi}(X)$ for each X.)*

Figure 5 (right) suggests an isomorphism $(A \times B)/R \to A/R \times B/R$ given by $[(v, w)] \mapsto ([v], [w])$. Indeed, this is a general principle, as is proved in Section 8.2 of [3]. Moreover, the figure suggests $[(v, w)] = [v] \times [w]$ (as sets). This too is true in general. Consequently, $|[(v, w)]| = |[v]| \cdot |[w]|$. The proof of our main theorem will employ these remarks.

5. The Cartesian Skeleton

We now recall the definition of the Cartesian skeleton $S(G)$ of an arbitrary graph G in Γ_0. The Cartesian skeleton $S(G)$ is a graph on the vertex set of G that has the property $S(A \times B) = S(A) \Box S(B)$ in the class of R-thin graphs, thereby linking the direct and Cartesian products.

We construct $S(G)$ as a certain subgraph of the Boolean square of G. The *Boolean square* of a graph G is the graph G^s with $V(G^s) = V(G)$ and $E(G^s) = \{xy \mid N_G(x) \cap N_G(y) \neq \emptyset\}$. Thus, xy is an edge of G^s whenever G has an x, y-walk of length two. For instance, if $p \geq 3$, then K_p^s is K_p with a loop added to each vertex. Also, $K_2^s = K_1^* + K_1^*$ and $K_1^s = K_1$. The left side of Figure 6 shows graphs A, B and $A \times B$ (bold) together with their Boolean squares A^s, B^s and $(A \times B)^s$ (dotted).

If G has an x, y-walk W of even length, then G^s has an x, y-walk of length $|W|/2$ on alternate vertices of W. Thus G^s is connected if G is connected and has an odd cycle. (The presence of an odd cycle guarantees an even walk between any two vertices of G.) On the other hand, if G is connected and bipartite, then G^s has exactly two components, and their respective vertex sets are the two partite sets of G.

We now show how to form $S(G)$ as a certain spanning subgraph of G^s. Given a factorization $G = A \times B$, we say that an edge $(a, b)(a', b')$ of G^s is

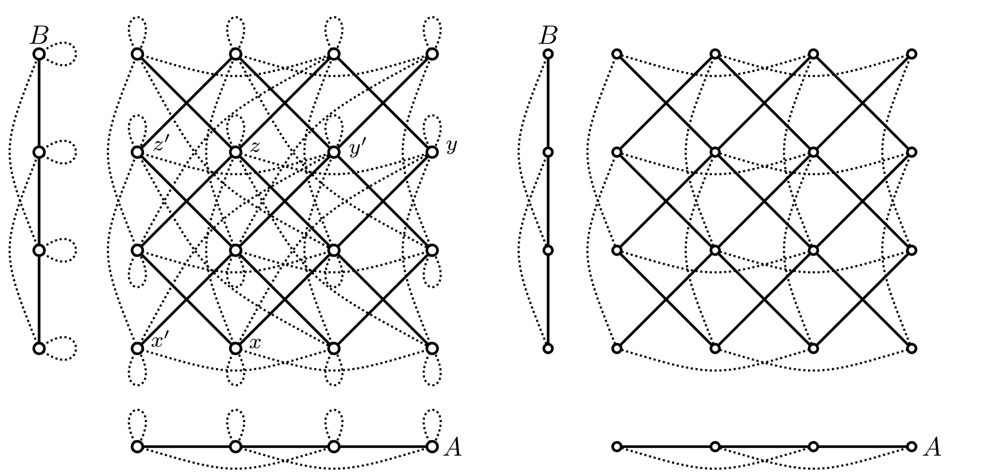


Figure 6: Left: Graphs $A, B, A \times B$ and their Boolean squares A^s, B^s and $(A \times B)^s$ (dotted). Right: Graphs $A, B, A \times B$ and their Cartesian skeletons $S(A), S(B)$ and $S(A \times B)$ (dotted).

Cartesian relative to the factorization $A \times B$ if either $a = a'$ and $b \neq b'$, or $a \neq a'$ and $b = b'$. For example, in Figure 6 edges xz and zy of G^s are Cartesian (relative to the factorization $A \times B$), but edges xy and yy of G^s are not Cartesian. Our goal is to form $S(G)$ from G^s by removing the edges of G^s that are not Cartesian, but we do this in a way that does not reference the factoring $A \times B$ of G. We identify three intrinsic criteria for edges of G^s that tell us if they may fail to be Cartesian relative to some factoring of G. (Note that the symbol $\subset$ means *proper* inclusion.)

(i) If xy is a loop (i.e. if $x = y$) then xy cannot be Cartesian.

(ii) In Figure 6 edge xy of G^s is not Cartesian, and there is a $z \in V(G)$ with $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ and $N_G(x) \cap N_G(y) \subset N_G(y) \cap N_G(z)$.

(iii) In Figure 6 the edge $x'y'$ of G^s is not Cartesian, and there is a $z' \in V(G)$ with $N_G(x') \subset N_G(z') \subset N_G(y')$.

Our aim is to remove from G^s all edges that meet one of these criteria. We package the above criteria into the following definition.

Definition 1. An edge xy of G^s is *dispensable* if $x = y$ or there exists $z \in V(G)$ for which both of the following statements hold.

(1) $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ or $N_G(x) \subset N_G(z) \subset N_G(y)$,

(2) $N_G(y) \cap N_G(x) \subset N_G(y) \cap N_G(z)$ or $N_G(y) \subset N_G(z) \subset N_G(x)$.

Observe that the above statements (1) and (2) are symmetric in x and y. Also, the second statement in (1) implies the first in (2), and the second in (2) implies the first in (1). With this in mind, we note that if either (ii) or (iii) above holds, then both conditions (1) and (2) in the definition hold.

Now we come to the main definition of this section.

Definition 2. The *Cartesian skeleton* of a graph G is the spanning subgraph $S(G)$ of G^s obtained by removing all dispensable edges from G^s.

The right side of Figure 6 is the same as its left side, except all dispensable edges of A^s, B^s and $(A \times B)^s$ are deleted. Thus the remaining dotted edges are $S(A), S(B)$ and $S(A \times B)$. Note that although $S(G)$ was defined without regard to the factorization $G = A \times B$, we nonetheless have $S(A \times B) = S(A) \Box S(B)$. The following proposition from [4] asserts that this always holds for R-thin graphs.

Proposition 2. *If A, B are R-thin graphs without isolated vertices, then $S(A \times B) = S(A) \Box S(B)$.*

We emphasize that by $S(A \times B) = S(A) \Box S(B)$ we mean *equality*, not mere isomorphism; the above graphs $S(A \times B)$ and $S(A) \Box S(B)$ have identical vertex and edge sets. As $S(G)$ is defined entirely in terms of the adjacency structure of G, we have the following immediate consequence of Definition 2.

Proposition 3. *Any isomorphism $\varphi : G \to H$, as a map $V(G) \to V(H)$, is also an isomorphism $\varphi : S(G) \to S(H)$.*

We will also need some results concerning connectivity of Cartesian skeletons. The following result (which does not require R-thinness) is from [4]. (For another proof, see Chapter 8 of [3].)

Proposition 4. *Suppose G is connected.*

(i) *If G has an odd cycle, then $S(G)$ is connected.*

(ii) *If G is bipartite, then $S(G)$ has two connected components. Their respective vertex sets are the two partite sets of G.*

6. The Proof of Theorem 1

We now prove Theorem 1, restated below for convenience.

Theorem 1. *Suppose A and B are connected graphs, and B is bipartite but A is not. If $A \times B$ admits an involution that reverses its partite sets, then B also admits an involution that reverses its partite sets.*

PROOF. Assume the hypotheses, and say B has partite sets B_0 and B_1. Take an involution $\varphi : A \times B \to A \times B$ that reverses the bipartition, that is, φ interchanges $V(A) \times B_0$ and $V(A) \times B_1$. Say $\varphi(v, w) = (\varphi_A(v, w), \varphi_B(v, w))$. As we will see, φ actually has a very simple expression. In what follows, we will reduce to R-thin graphs so that we can apply the Cartesian skeleton operator. We will then show that B/R admits an involution that reverses its partite sets; from there we will argue that B does too.

As noted in Section 4, the map $\varphi : A \times B \to A \times B$ induces a map

$$\widetilde{\varphi} : (A \times B)/R \to (A \times B)/R,$$

where $\widetilde{\varphi}([(v,w)] = [\varphi(v,w)] = [(\varphi_A(v,w), \varphi_B(v,w))]$. Because $(A \times B)/R \cong A/R \times B/R$ by the isomorphism $[(a,b)] \mapsto ([a],[b])$, we can view $\widetilde{\varphi}$ as a map

$$\begin{array}{rcl} \widetilde{\varphi} : A/R \times B/R & \to & A/R \times B/R \\ ([v],[w]) & \mapsto & ([\varphi_A(v,w)], [\varphi_B(v,w)]). \end{array}$$

Now, A/R is connected and nonbipartite because any x,y-walk in the connected graph A induces an $[x]$,$[y]$-walk in A/R of the same length (possibly with loops); and thus a closed odd walk in A induces a closed odd walk in A/R. By definition of R, any R-equivalence class in B is clearly contained entirely in one of the partite sets B_0 or B_1. Thus B/R is connected and bipartite. It follows that the above map $\widetilde{\varphi} : A/R \times B/R \to A/R \times B/R$ is a bipartition-reversing involution on R-thin graphs.

Applying the Cartesian skeleton operator and using Proposition 3, this is an involution

$$\widetilde{\varphi} : S(A/R \times B/R) \to S(A/R \times B/R),$$

and we get the upper-most square in the following Commutative Diagram (2). The remainder of the diagram proceeds as follows. Proposition 2 applied to the second line yields the third line. (The vertical double lines indicate equality and the horizontal arrows are involutions.)

$$\begin{array}{ccc} A/R \times B/R & \xrightarrow{\widetilde{\varphi}} & A/R \times B/R \\ S\big\downarrow & & S\big\downarrow \\ S(A/R \times B/R) & \xrightarrow{\widetilde{\varphi}} & S(A/R \times B/R) \\ \| & & \| \\ S(A/R)\Box S(B/R) & \xrightarrow{\widetilde{\varphi}} & S(A/R)\Box S(B/R) \\ \| & & \| \\ S(A/R)\Box(B'_0 + B'_1) & \xrightarrow{\widetilde{\varphi}} & S(A/R)\Box(B'_0 + B'_1) \\ \| & & \| \\ S(A/R)\Box B'_0 + S(A/R)\Box B'_1 & \xrightarrow{\widetilde{\varphi}} & S(A/R)\Box B_0 + S(A/R)\Box B'_1 \end{array} \qquad (2)$$

For the fourth line, note that the boolean square of the bipartite graph B/R consists of two connected components whose respective vertex sets are the partite sets of B/R. In turn, the skeleton $S(B/R)$ consists of two connected components B'_0 and B'_1 whose respective vertex sets are the partite sets of

B/R. (See Proposition 4.) This gives the fourth line of Diagram (2). The distributive property of $\Box$ gives the bottom line.

Consider the involution $\widetilde{\varphi}$ on the bottom line of Diagram (2). The restrictions of $\widetilde{\varphi}$ to the two components $S(A/R)\Box B_0'$ and $S(A/R)\Box B_1'$ give us two isomorphisms, as follows.

$$\begin{array}{c} S(A/R)\Box B_1' \xrightarrow{\widetilde{\varphi}} S(A/R)\Box B_0' \\ S(A/R)\Box B_0' \xrightarrow{\widetilde{\varphi}} S(A/R)\Box B_1' \end{array} \tag{3}$$

Because $\widetilde{\varphi}$ is an involution, these two restrictions are inverses of one another. Make note that as $V(A/R) = V(S(A/R))$ and $V(B/R) = V(B_0') + V(B_1')$, the map $\widetilde{\varphi}$ is an involution of both $A/R \times B/R$ and $S(A/R)\Box S(B/R)$.

Identify $S(A/R)\Box B_1'$ with its prime factoring over the Cartesian product, by the following isomorphism:

$$\begin{array}{rcl} S(A/R)\Box B_1' & \to & A_1\Box A_2\Box\cdots\Box A_k \,\Box\, B_{11}\Box B_{12}\Box\cdots\Box B_{1\ell} \\ (a,b) & \mapsto & (a_1,a_2,\ldots,a_k,\ b_{11},b_{12},\ldots,b_{1\ell}). \end{array} \tag{4}$$

Here the A_i are the prime factors of $S(A/R)$ and the B_{1i} are the prime factors of B_1'. (Thus the a_i are functions of vertices $a \in V(S(A/R)$ and the b_{1i} are functions of vertices $b \in B_1'$.) Although this is an isomorphism between two graphs, to avoid unnecessary notation we view it as a relabeling of the vertices of $S(A/R)\Box B_1'$ by the association $(a,b) \mapsto (a_1,a_2,\ldots,a_k,\ \ b_{11},b_{12},\ldots,b_{1\ell})$. We thus identify the two graphs in the isomorphism (4).

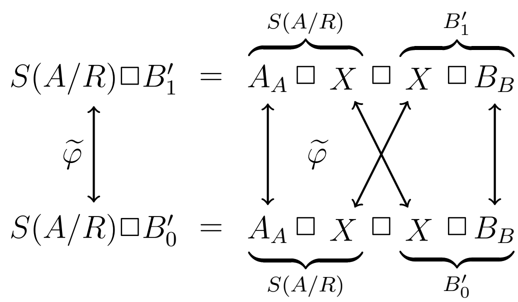
Similarly, there is an identification

$$\begin{array}{rcl} S(A/R)\Box B_0' & = & A_1\Box A_2\Box\cdots\Box A_k \,\Box\, B_{01}\Box B_{02}\Box\cdots\Box B_{0\ell} \\ (a,b) & \mapsto & (a_1,a_2,\ldots,a_k,\ b_{01},b_{02},\ldots,b_{0\ell}). \end{array} \tag{5}$$

By unique prime factoring over $\Box$, the B_{0i} are isomorphic to the B_{1i}, up to order. Combining isomorphisms (3), (4) and (5), we have:

$$\begin{array}{ccl} S(A/R)\Box B_1' & \longrightarrow & (A_1\Box A_2\Box\cdots\Box A_k) \,\Box\, (B_{11}\Box B_{12}\Box\cdots\Box B_{1\ell}) \\ \widetilde{\varphi}\updownarrow & & \\ S(A/R)\Box B_0' & \longrightarrow & (A_1\Box A_2\Box\cdots\Box A_k) \,\Box\, (B_{01}\Box B_{02}\Box\cdots\Box B_{0\ell}). \end{array} \tag{6}$$

By Corollary 1, we can identify the prime factors in the upper row of Diagram (6) with those in the bottom row, so that the involution $\widetilde{\varphi}$ simply permutes its arguments by some permutation π (in this case, of order 2). Now, π may send some prime factors of $S(A/R)$ (on the bottom row) to factors of B_1' (on the upper row). Let X be the product of the factors of $S(A/R)$ that π sends to factors of B_1'. Then there is a corresponding product X of factors of B_0' that π sends to factors of $S(A/R)$. Using commutativity of the Cartesian product, we regroup the prime factors so that $S(A/R) \cong A_A \Box X$ and $B_0' \cong B_1' \cong X \Box B_B$. (Thus A_A stands for the product of factors of $S(A/R)$ sent to $S(A/R)$, and B_B stands for the product of factors of B_0' sent to B_1'.) This scheme is indicated in Diagram (7) below.



(7)

We have now relabeled the vertices of $S(A/R)$ with ordered pairs $(a, x) \in V(A_A \Box X)$, and those of B_0' and B_1' with pairs $(x, b) \in V(X \Box B_B)$, respectively. With this labeling, the involution $\widetilde{\varphi}$ has the particularly simple form

$$\widetilde{\varphi}((a, x), (y, b)) = ((a, y), (x, b)). \tag{8}$$

Recall that $V(A/R) = V(S(A/R))$, and that the two partite sets of B/R are the sets $V(B_0')$ and $V(B_1')$. Thus $A/R \times B/R$ has partite sets $V(S(A/R)) \times V(B_0')$ and $V(S(A/R)) \times V(B_1')$, that is, these partite sets are the vertex sets of the two graphs on the left column of Diagram (7). In this diagram, the vertices of one partite set of $A/R \times B/R$ are labeled with vertices of $(A_A \Box X) \Box (X \Box B_B)$, and these appear at the top of the diagram. Likewise, the vertices of the other partite set are also labeled by the vertices of $(A_A \Box X) \Box (X \Box B_B)$, and they appear on the bottom of the diagram. Further, one partite set of B/R is labeled as $V(B_1') = V(X \Box B_B)$, and the other partite set is labeled as $V(B_0') = V(X \Box B_B)$.

There is some danger of confusion here because each vertex $(y, b) \in V(X \Box B_B)$ simultaneously labels both a vertex in the partite set $V(B_0')$ of

B/R *and* a vertex in the other partite set $V(B_1')$. As a bookkeeping device, when (y,b) denotes a vertex in $V(B_1')$, we write is as $\overline{(y,b)}$. If it denotes a vertex in $V(B_0')$, then we write it simply as (y,b). Thus the bar has no significance other than an indication of which partite set the pair (y,b) belongs to. But we can regard the map $(y,b) \mapsto \overline{(y,b)}$ as a bijection from one partite set of B/R to the other partite set. With this slight adjustment of notation, the involution (8) can be updated as

$$\begin{aligned} \widetilde{\varphi}((a,x),(y,b)) &= \left((a,y),\overline{(x,b)}\right), \text{ and} \\ \widetilde{\varphi}\left((a,x),\overline{(y,b)}\right) &= (a,y),(x,b)). \end{aligned} \tag{9}$$

Also define the following order-2 permutation $\mu : V(B/R) \to V(B/R)$ that reverses the partite sets of B/R:

$$\mu((y,b)) = \overline{(y,b)} \quad \text{and} \quad \mu\left(\overline{(y,b)}\right) = (y,b).$$

We next show that μ is an automorphism (hence involution) of B/R. Suppose $(y,b)\overline{(y',b')} \in E(B/R)$. Applying μ yields the pair $\overline{(y,b)}(y',b')$, and we must show that this is also an edge of B/R.

Select an edge $(a,x)(a',x') \in E(A/R)$. We thus have

$$((a,x),(y,b))\left((a',x'),\overline{(y',b')}\right) \in E(A/R \times B/R). \tag{10}$$

Applying the involution $\widetilde{\varphi}$ (Equation (9)) yields

$$\left((a,y),\overline{(x,b)}\right)((a',y'),(x',b')) \in E(A/R \times B/R). \tag{11}$$

From this, $(a,y)(a',y') \in E(A/R)$. Because $(y,b)\overline{(y',b')} \in E(B/R)$, we get

$$((a,y),(y,b))\left((a',y'),\overline{(y',b')}\right) \in E(A/R \times B/R). \tag{12}$$

Applying $\widetilde{\varphi}$ to this,

$$\left((a,y),\overline{(y,b)}\right)((a',y'),(y',b')) \in E(A/R \times B/R). \tag{13}$$

From this last edge of $A/R \times B/R$, we see that indeed $\overline{(y,b)}(y',b') \in E(B/R)$, and hence μ is a bijective homomorphism from B/R to itself. As $\mu = \mu^{-1}$, we see that μ is an involution of B/R (that reverses partite sets).

Now that we have produced an involution μ of B/R that interchanges partite sets, we will lift it to an involution of B. Recall that we began the proof with an involution $\varphi : A \times B \to A \times B$ that reversed partite sets of $A \times B$. This induced an involution $\widetilde{\varphi} : (A \times B)/R \to (A \times B)/R$, where $\widetilde{\varphi}([(v,w)]) = [\varphi(v,w)] = [(\varphi_A(v,w), \varphi_B(v,w))]$. By the isomorphism

$$\begin{array}{rcl} (A \times B)/R & \to & A/R \times B/R \\ [(v,w)] & \mapsto & ([v],[w]), \end{array}$$

this became the involution

$$\begin{array}{rcl} \widetilde{\varphi} : A/R \times B/R & \to & A/R \times B/R \\ ([v],[w]) & \mapsto & ([\varphi_A(v,w)], [\varphi_B(v,w)]). \end{array}$$

We then relabeled vertices as $[v] = (a,x)$ and $[w] = (y,b)$ or $\overline{(y,b)}$. (So (a,x) and (y,b) actually denote R-equivalence classes.) Under this relabeling, $\widetilde{\varphi}$ became

$$\widetilde{\varphi}((a,x),(y,b)) \;=\; \Big((a,y), \overline{(x,b)}\Big).$$

Regard the equivalence class $[(v,w)] \in V((A \times B)/R)$ as a subset of $V(A \times B)$, and recall the remarks in Section 4. Then $[(v,w)] = [v] \times [w] = (a,x) \times (y,b)$, where $\times$ indicates the Cartesian product of sets. The involution φ sends the subset $[(v,w)] = (a,x) \times (y,b)$ of the partite set B_0 of $A \times B$ bijectively to $[\varphi(v,w)] = [(\varphi_A(v,w), \varphi_B(v,w))] = [\varphi_A(v,w)] \times [\varphi_B(v,w)] = (a,y) \times \overline{(x,b)}$ in the other partite set of $A \times B$. Thus $|(a,x) \times (y,b)| = |(a,y) \times \overline{(x,b)}|$, so

$$|(a,x)| \cdot |(y,b)| = |(a,y)| \cdot \left|\overline{(x,b)}\right|.$$

For $x = y$, this is

$$|(a,y)| \cdot |(y,b)| = |(a,y)| \cdot \left|\overline{(y,b)}\right|.$$

From this, $|(y,b)| = \left|\overline{(y,b)}\right| = |\mu(y,b)|$. Therefore our involution μ of B/R preserves the cardinalities of R-equivalence classes that make up the vertices of B/R. By Proposition 1, we can thus extend it to a (partition-reversing) involution of B by declaring it to be a bijection on the R-equivalence classes of B/R. This completes the proof. ■

This also completes the proof of Theorem 2 in Section 2.

7. All Prime Factorings

As noted in the introduction, connected bipartite graphs do not have unique prime factorings. However, we have proved that any factor of K_2 is unique. We now describe how to find all prime factorings of a connected graph that has a K_2 factor. The results of this section hinge on the ideas laid out in the article [2], and summarized here.

In what follows, let $G \cong A \times K_2$ be a connected graph with K_2 as a prime factor. Then G is bipartite and admits an involution that reverses its partite sets X and Y, because $A \times K_2$ these properties.

Of course there may be many involutions σ that interchange X and Y. Given such a σ, we define the graph G^σ as follows.

$$\begin{aligned} V(G^\sigma) &= \{\{x, \sigma(x)\} \mid x \in V(G)\} \\ E(G^\sigma) &= \{\{x, \sigma(x)\}\{y, \sigma(y)\} \mid xy \in E(G)\}. \end{aligned}$$

As in the proof of Lemma 1, we have $G \cong G^\sigma \times K_2$. Because G is connected, it follows that G^σ is connected and nonbipartite, for otherwise $G = G^\sigma \times K_2$ would have more than one component.

Let $\mathrm{RI}(G)$ be the set of involutions of G that reverse its bipartition. The group $\mathrm{Aut}(G)$ acts on $\mathrm{RI}(G)$ by conjugation. Choose representatives $\sigma_1, \sigma_2, \ldots, \sigma_k$ of the conjugacy classes. As proved in [2], the set of isomorphism classes of graphs A for which $G \cong A \times K_2$ is precisely

$$A = G^{\sigma_1}, G^{\sigma_2}, \ldots, G^{\sigma_k}. \tag{14}$$

Thus the distinct factorings of G as $G \cong A \times K_2$ are precisely

$$G^{\sigma_1} \times K_2, \quad G^{\sigma_2} \times K_2, \; \ldots, \; G^{\sigma_k} \times K_2.$$

The results of the present paper imply that the K_2 factor is unique; therefore any prime factoring of G is a refinement of one of the above $G^{\sigma_i} \times K_2$ by a prime factoring of G^{σ_i}. Since the class of connected nonbipartite graphs obeys unique factorization (as noted in the introduction), each G^{σ_i} has a distinct prime factorization. These factorizations can be found (say) with Algorithm 24.7 in Chapter 24 of [3]. Carrying out these factorizations for each G^{σ_i} in List (14) above, we obtain every distinct prime factoring of G.

In fact, if Conjecture 1 of Section 1 is true, then the above reasoning can be extended to compute every prime factoring of any connected bipartite graph. To see this, we will use a standard result due to Lovász [6].

Proposition 5. *If $A \times B \cong A' \times B$ and there is a homomorphism $K \to B$, then $A \times K \cong A' \times K$.*

As any nontrivial bipartite graph B admits homomorphisms $K_2 \to B$ and $B \to K_2$, it follows that $A \times B \cong A' \times B$ if and only if $A \times K_2 \cong A' \times K_2$.

Now suppose Conjecure 1 is true. Let G be any connected bipartite graph, and say that B is its unique prime bipartite factor. Select a factoring

$$G \cong A \times B$$

of G, and let

$$H = A \times K_2.$$

By the previous paragraph, the various factorings $G \cong G' \times B$ are in one-to-one correspondence with the factorings $H \cong G' \times K_2$. As above, we can compute from $H \cong G' \times K_2$ all G' and their prime factorings. Appending B to each of these factorings, we obtain all prime factorings of G. (Observe that to carry out this process, it is necessary to first find a factoring $G \cong A \times B$.)

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