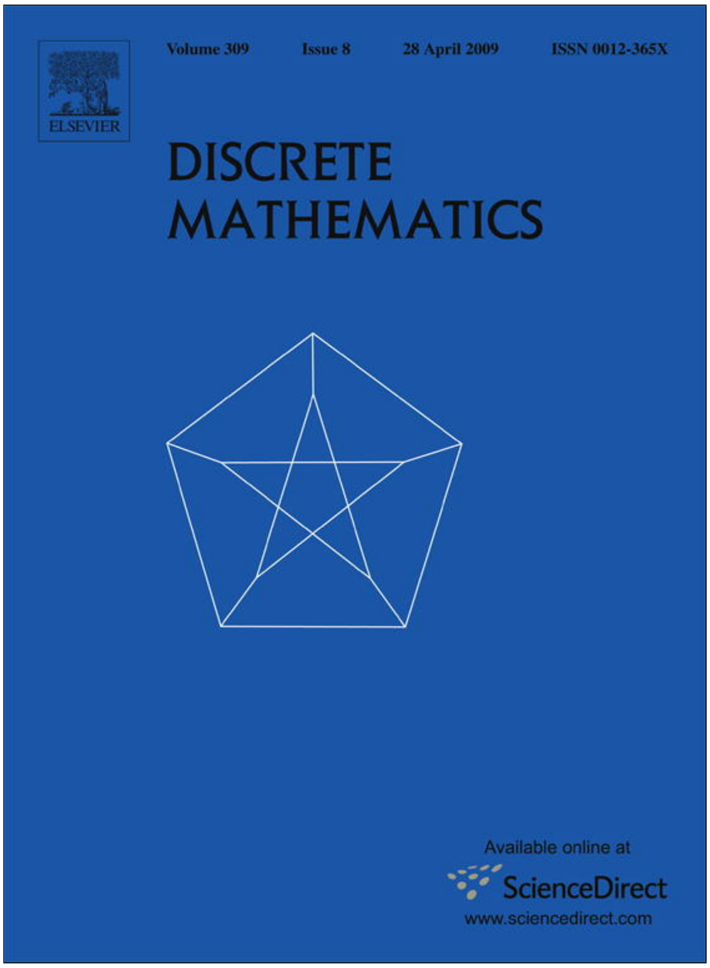
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Note

On direct product cancellation of graphs

Richard H. Hammack

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA 23284-2014, USA

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ABSTRACT

The direct product of graphs obeys a limited cancellation property. Lovász proved that if C has an odd cycle then $A \times C \cong B \times C$ if and only if $A \cong B$, but cancellation can fail if C is bipartite. This note investigates the ways cancellation can fail. Given a graph A and a bipartite graph C, we classify the graphs B for which $A \times C \cong B \times C$. Further, we give exact conditions on A that guarantee $A \times C \cong B \times C$ implies $A \cong B$. Combined with Lovász's result, this completely characterizes the situations in which cancellation holds or fails.

1. Introduction

Recently there has been a revival of interest in questions involving cancellation properties of various graph products. The articles [1–3,5] investigate sufficient conditions under which $A \star C \cong B \star C$ implies $A \cong B$, where A, B and C are graphs, and $\star$ stands for either the Cartesian product, the strong product, or the direct product. In this contribution we give a complete solution to the cancellation problem for the direct product.

For us, a *graph* A is a symmetric binary relation $E(A)$ on a finite set $V(A)$ of vertices. We call elements of $E(A)$ *edges* and denote them as aa', where $a, a' \in V(A)$; reflexive elements aa are called *loops*. The *direct product* of two graphs A and B is the graph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose edges are the pairs $(a, b)(a', b')$ with $aa' \in E(A)$ and $bb' \in E(B)$. (See [4] for a standard reference.) A *homomorphism* from graph A to graph B is a map $\varphi : V(A) \to V(B)$ with the property that $aa' \in E(A)$ implies $\varphi(a)\varphi(a') \in E(B)$. We are indebted to Lovász for the following theorems.

Theorem 1 (Lovász [6], Theorem 6). *Let A, B, C and D be graphs. If $A \times C \cong B \times C$ and there is a homomorphism from D to C, then $A \times D \cong B \times D$.*

Theorem 2 (Lovász [6], Theorem 7). *Let A, B and C be graphs. If $A \times C \cong B \times C$, then there is an isomorphism from $A \times C$ to $B \times C$ of the form $(a, c) \mapsto (\psi(a, c), c)$ for some homomorphism $\psi : A \times C \to B$.*

Theorem 3 (Lovász [6], Theorem 9). *Let A, B and C be graphs. If C has an odd cycle, then $A \times C \cong B \times C$ if and only if $A \cong B$.*

Theorem 3 can be interpreted as a partial cancellation law, as it gives sufficient conditions under which the common factor C can be "cancelled" from the expression $A \times C \cong B \times C$. The theorem is quite strong in the sense that cancellation can always fail if C is bipartite. Indeed, as Lovász observed, if C fails to have an odd cycle, then there exist graphs A and B for which $A \times C \cong B \times C$ but $A \ncong B$. Fig. 1(a) and (b) show simple examples, where, in each case, C is the complete graph K_2. In Fig. 1(a), A is K_3 and B is a path of length 2 with loops at each end, while $A \times C$ and $B \times C$ are both isomorphic to the 6-cycle. In Fig. 1(b), $A \times C$ and $B \times C$ both consist of two disjoint 4-cycles, but $A \ncong B$.

However, Theorem 3 does not completely resolve the question of when C can be cancelled from $A \times C \cong B \times C$. Although it does imply that cancellation can fail if and only if C is bipartite, it does not address what properties of A (or B) might

E-mail address: rhammack@vcu.edu.

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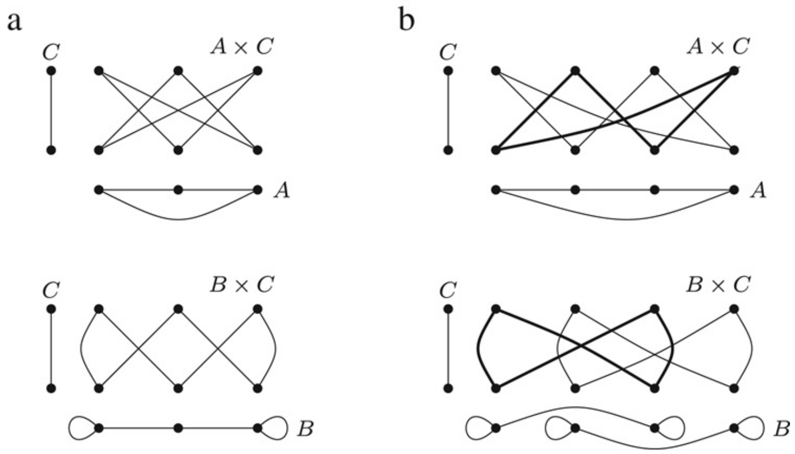

Fig. 1. Failure of cancellation.

guarantee that cancellation holds. For example, if A consists of a single vertex with a loop, then surely $A \times C \cong B \times C$ implies $A \cong B$, whether or not C is bipartite. We might reasonably ask what other graphs A have this property.

The present note answers that question. Given a graph A and a bipartite graph C, we classify those graphs B for which $A \times C \cong B \times C$. This leads to exact conditions on A which guarantee that $A \times C \cong B \times C$ implies $A \cong B$.

Our methods involve two new ideas. Section 2 introduces the notion of an *anti-automorphism* of a graph, and Section 3 describes a "factorial" operation on graphs. We combine these constructions in Section 4 to answer our main questions.

We note in passing that a standard (but difficult) result states that the class of connected non-trivial non-bipartite graphs obeys unique factorization with respect to the direct product [4,7]. Given this, it is immediate that $A \times C \cong B \times C$ if and only if $A \cong B$ when all factors are connected, non-bipartite and non-trivial. However, Theorem 3 (and our main theorems) are more general in the sense that connectivity is not assumed and A and B are not required to have odd cycles.

2. Anti-automorphisms

An **anti-automorphism** of a graph A is a bijection $\mu : V(A) \to V(A)$ with the property that $aa' \in E(A)$ if and only if $\mu(a)\mu^{-1}(a') \in E(A)$ for all pairs $a, a' \in V(A)$. The set of all anti-automorphisms of A is denoted $\text{Ant}(A)$.

In general, the set $\text{Ant}(A)$ is not a group, though it contains the identity and is closed with respect to taking inverses. Notice that any automorphism of order 2 is an anti-automorphism. The following construction will be of key importance in this article.

Given an anti-automorphism μ of a graph A, we define a graph A^μ as $V(A^\mu) = V(A)$ and $E(A^\mu) = \{a\mu(a') : aa' \in E(A)\}$. For example, let $A = K_3$ and μ be a transposition of two vertices (which is an automorphism of order 2, and thus an anti-automorphism). Then A^μ is a path of length 2 with loops at each end. Thus in Fig. 1(a), we have $B = A^\mu$. Similarly, $B = A^\mu$ in Fig. 1(b), where μ is reflection of A across the vertical axis.

We take care to point out that the statement $aa' \in E(A) \Leftrightarrow a\mu(a') \in E(A^\mu)$ is true, and it follows not just from the definition of A^μ, but also from the fact that μ is an anti-automorphism. This is summarized in the following result, which will be used frequently and without further comment.

Proposition 4. *If $\mu \in \text{Ant}(A)$, then $aa' \in E(A)$ if and only if $a\mu(a') \in E(A^\mu)$.*

Proof. Certainly if $aa' \in E(A)$, then $a\mu(a') \in E(A^\mu)$ by definition of A^μ. Conversely, suppose $a\mu(a') \in E(A^\mu)$. By definition of A^μ, this means that either $aa' \in E(A)$ or $\mu^{-1}(a)\mu(a') \in E(A)$. In the second case, the fact that μ is an anti-automorphism ensures that $aa' \in E(A)$. ■

The fact that $B = A^\mu$ in Fig. 1(a) and (b) illustrates the following general principle.

Proposition 5. *Let A and B be graphs. If C is a bipartite graph that has at least one edge, then $A \times C \cong B \times C$ if and only if $B \cong A^\mu$ for some $\mu \in \text{Ant}(A)$.*

Proof. Suppose $A \times C \cong B \times C$. We will construct an anti-automorphism μ of A for which $A^\mu \cong B$. Since C has an edge, there is a homomorphism $K_2 \to C$, and therefore Theorem 1 implies $A \times K_2 \cong B \times K_2$. By Theorem 2, there is an isomorphism $A \times K_2 \to B \times K_2$ of form $(a, c) \mapsto (\psi(a, c), c)$. Put $V(K_2) = \{0, 1\}$ and define maps $\alpha, \beta : V(A) \to V(B)$ as follows.

$$\alpha(a) = \psi(a, 0)$$
$$\beta(a) = \psi(a, 1).$$

Since $(a, c) \mapsto (\psi(a, c), c)$ is an isomorphism, it follows readily that α and β are bijective. We now show that the composition $\alpha^{-1}\beta$ is an anti-automorphism. Observe that

$$\begin{aligned} aa' \in E(A) &\iff (a, 0)(a', 1) \in E(A \times K_2) \\ &\iff (\psi(a, 0), 0)(\psi(a', 1), 1) \in E(B \times K_2) \\ &\iff (\alpha(a), 0)(\beta(a'), 1) \in E(B \times K_2) \\ &\iff \alpha(a)\beta(a') \in E(B). \end{aligned}$$

Thus we have

$$aa' \in E(A) \iff \alpha(a)\beta(a') \in E(B), \tag{1}$$

and from this it follows that also $bb' \in E(B) \iff \beta^{-1}(b)\alpha^{-1}(b') \in E(A)$. Therefore

$$\begin{aligned} aa' \in E(A) &\iff \alpha(a)\beta(a') \in E(B) \\ &\iff \beta^{-1}\alpha(a)\alpha^{-1}\beta(a') \in E(A) \\ &\iff (\alpha^{-1}\beta)^{-1}(a)\,\alpha^{-1}\beta(a') \in E(A). \end{aligned}$$

This means $\alpha^{-1}\beta \in \mathrm{Ant}(A)$. Set $\mu = \alpha^{-1}\beta$. Notice that $\alpha : A^{\mu} \to B$ is an isomorphism: By definition, any edge of A^{μ} has the form $a\mu(a') = a\alpha^{-1}\beta(a')$ for some $aa' \in V(A)$. Taking α of both endpoints produces the edge $\alpha(a)\beta(a')$, which by (1) is an edge of B. On the other hand, if $bb' \in E(B)$, then $\alpha^{-1}(b)\beta^{-1}(b') \in E(A)$, so $\alpha^{-1}(b)\mu\beta^{-1}(b') \in E(A^{\mu})$, which reduces to $\alpha^{-1}(b)\alpha^{-1}(b') \in E(A^{\mu})$. Therefore $B \cong A^{\mu}$.

Conversely, it suffices to prove that $A \times C \cong A^{\mu} \times C$ for any bipartite graph C and $\mu \in \mathrm{Ant}(A)$. Let C_0 and C_1 be a bipartition of C, and define a map $\Theta : A \times C \to A^{\mu} \times C$ as

$$\Theta(a, c) = \begin{cases} (a, c) & \text{if } c \in C_0 \\ (\mu(a), c) & \text{if } c \in C_1. \end{cases}$$

This is clearly bijective. Suppose $(a, c)(a', c') \in E(A \times C)$. We may assume $c \in C_0$ and $c' \in C_1$. Then $\Theta(a, c)\Theta(a', c') = (a, c)(\mu(a'), c') \in E(A^{\mu} \times C)$. In the other direction, any edge of $A^{\mu} \times C$ must be either of form $(a, c)(\mu(a'), c')$ or $(\mu(a), c)(a', c')$, where in each case $c \in C_0, c' \in C_1$ and $aa' \in E(A)$. In the first case, $(a, c)(\mu(a'), c')$ is the image under Θ of the edge $(a, c)(a', c')$ of $A \times C$. In the second case, $(\mu(a), c)(a', c')$ is the image under Θ of $(\mu(a), c)(\mu^{-1}(a'), c')$, which is an edge of $A \times C$ because μ is an anti-automorphism. ■

Proposition 5 implies that the set $\mathrm{Ant}(A)$ in some sense parameterizes the graphs B for which $A \times C \cong B \times C$. For any $\mu \in \mathrm{Ant}(A)$, the graph $B = A^{\mu}$ satisfies $A \times C \cong B \times C$. Conversely for any B with $A \times C \cong B \times C$, there is some $\mu \in \mathrm{Ant}(A)$ for which $B \cong A^{\mu}$. However, this correspondence needn't be injective. There can exist distinct anti-automorphisms μ and λ for which $A^{\mu} \cong A^{\lambda}$. For example, if $A = K_3$, there are three distinct transpositions μ_1, μ_2 and μ_3 that interchange two vertices and fix the third. Each is an anti-automorphism, and $A^{\mu_1} \cong A^{\mu_2} \cong A^{\mu_3}$ is the path of length 2 with loops at each end. As a tool for sorting out which anti-automorphism yield isomorphic graphs, we introduce the notion of a graph factorial.

3. A graph factorial

Here we define an operation on graphs that mimics the factorial of a positive integer.

The **factorial** of a graph A is the graph, denoted $A!$, whose vertices are the permutations of $V(A)$. Permutations λ and μ are adjacent in $A!$ exactly when $aa' \in E(A) \Leftrightarrow \lambda(a)\mu^{-1}(a') \in E(A)$ for all pairs $a, a' \in V(A)$. We denote an edge joining vertices λ and μ as $(\lambda)(\mu)$ in order to avoid confusion with composition.

Notice that $A!$ is well-defined as a symmetric graph since replacing a and a' in the definition with $\lambda^{-1}(a)$ and $\mu(a')$ yields $\lambda^{-1}(a)\mu(a') \in E(A) \Leftrightarrow aa' \in E(A)$.

Observe that there is a loop at a vertex μ of $A!$ if and only if $\mu \in \mathrm{Ant}(A)$. Also, if μ is an automorphism of A, then $(\mu)(\mu^{-1}) \in E(A!)$ but not every edge of $A!$ necessarily has this form. As an example of a graph factorial, let K_p^* be the complete graph on p vertices with loops at each vertex. Then any pair of permutations of $V(K_p^*)$ must be adjacent in $K_p^*!$, so $K_p^*! \cong K_{p!}^*$. Consequently

$$K_p^*! \;\cong\; K_p^* \times K_{p-1}^* \times K_{p-2}^* \times \cdots \times K_3^* \times K_2^*.$$

Of course we expect no such nice formulas for $A!$ when A is arbitrary.

Fig. 2(a) and (b) illustrate factorials of two graphs on the vertices $\{1, 2, 3\}$. In each case, id is the identity permutation, μ_i is the transposition of the two vertices $\{1, 2, 3\} - \{i\}$, and ρ_1 and ρ_2 are clockwise rotations of $2\pi/3$ and $4\pi/3$.

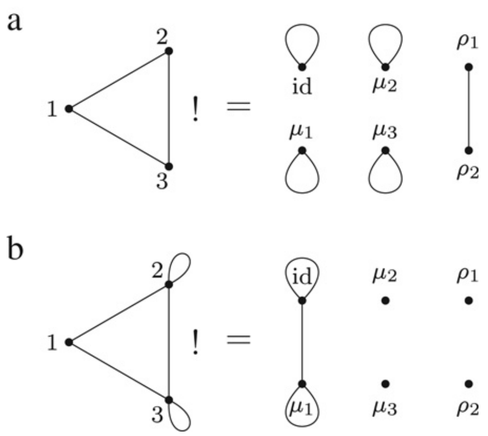

Fig. 2. Factorials of some graphs.

Proposition 6. *For any graph A, each non-trivial component of A! either is K_p^* for some p or is a complete bipartite graph.*

Proof. We first prove by induction that given any odd walk $(\mu_1)(\mu_2)(\mu_3)\dots(\mu_{2p})$ in $A!$, the pair $(\mu_1)(\mu_{2p})$ is an edge of $A!$. This is trivial if $p = 1$. If $p > 1$, the induction hypothesis guarantees $(\mu_3)(\mu_{2p}) \in E(A!)$, so $(\mu_1)(\mu_2)(\mu_3)(\mu_{2p})$ is a walk in $E(A!)$. Using the fact that the edges of this walk are edges in $A!$, we get

$$\begin{aligned} aa' \in E(A) &\iff \mu_1(a)\mu_2^{-1}(a') \in E(A) \\ &\iff \mu_3^{-1}\mu_1(a)\mu_2\mu_2^{-1}(a') \in E(A) \\ &\iff \mu_3\mu_3^{-1}\mu_1(a)\mu_{2p}^{-1}\mu_2\mu_2^{-1}(a') \in E(A) \\ &\iff \mu_1(a)\mu_{2p}^{-1}(a') \in E(A). \end{aligned}$$

Therefore $(\mu_1)(\mu_{2p}) \in E(A!)$.

Now, if C is a component of $A!$ that happens to be bipartite, then there is an odd path between any vertices α and β that are in different partite sets of C. Thus $(\alpha)(\beta) \in E(A!)$, so C is a complete bipartite graph. On the other hand, if C has an odd cycle (possibly just a loop), then there is an odd walk joining any pair of its vertices, so all pairs of vertices in C are adjacent, so $C \cong K_p^*$. ■

Since anti-automorphisms of A correspond to loops in $A!$, and since Proposition 6 implies that any component of $A!$ with a loop is isomorphic to a K_p^*, it follows that $\text{Ant}(A)$ is the set of all vertices belonging to the K_p^* components of $A!$. The next proposition shows that these components have a special significance.

Proposition 7. *If λ and μ are anti-automorphisms in the same component of $A!$, then $A^\lambda = A^\mu$.*

Proof. An arbitrary edge of A^λ has form $a\lambda(a')$ where aa' is an appropriate edge of A. Since λ and μ are adjacent in $A!$, it follows that $\mu^{-1}(a)\lambda(a') \in E(A)$. Therefore $a\lambda(a') = \mu(\mu^{-1}(a))\lambda(a')$ is an edge of A^μ. Thus every edge of A^λ is also an edge of A^μ. Reversing the roles of λ and μ, every edge of A^μ is an edge of A^λ. ■

As an example of this result, consider Fig. 2(b). There id and μ_1 belong to a K_2^* and it is easy to check that $A = A^{\text{id}} = A^{\mu_1}$. But despite Proposition 7, if anti-automorphisms λ and μ are in different components of $A!$, then this by itself says nothing about the relationship between A^λ and A^μ. For example, in Fig. 2(a) we have $A = A^{\text{id}} \not\cong A^{\mu_1} \cong A^{\mu_2} \cong A^{\mu_3}$. In the next section we resolve this issue by introducing an equivalence relation on $\text{Ant}(A)$ that is finer than the relation of belonging to the same K_p^* in $A!$.

4. Cancellation theorems

Given a graph A, we define a relation $\simeq$ on $\text{Ant}(A)$ by declaring $\mu \simeq \lambda$ if $\mu = \alpha\lambda\beta$ for some edge (possibly a loop) $(\alpha)(\beta) \in E(A!)$. Observe that this is an equivalence relation. It is reflexive because $\mu = \text{id}\,\mu\,\text{id}$. It is symmetric, for given that $\mu \simeq \lambda$, we have $\mu = \alpha\lambda\beta$ for $(\alpha)(\beta) \in E(A!)$. But then $\lambda = \alpha^{-1}\mu\beta^{-1}$, and $(\alpha^{-1})(\beta^{-1}) \in E(A!)$, so $\lambda \simeq \mu$. To check transitivity, suppose $\mu \simeq \lambda$ and $\lambda \simeq \kappa$. Then $\mu = \alpha\lambda\beta$ and $\lambda = \gamma\kappa\delta$ for edges $(\alpha)(\beta)$ and $(\gamma)(\delta)$ in $E(A!)$, so $\mu = \lambda\gamma\,\kappa\,\delta\beta$. But $(\alpha\gamma)(\delta\beta) \in E(A!)$ because $aa' \in E(A) \Leftrightarrow \gamma(a)\delta^{-1}(a') \in E(A) \Leftrightarrow \alpha\gamma(a)\beta^{-1}\delta^{-1}(a') \in E(A) \Leftrightarrow \alpha\gamma(a)(\delta\beta)^{-1}(a') \in E(A)$. Therefore $\mu \simeq \kappa$.

As an example, let us compute the equivalence classes for the case $A = K_3$. The graphs A and $A!$ are shown in Fig. 2(a). Consider the equivalence class containing μ_1. Since every edge (or loop) of $A!$ has as endpoints permutations that are both odd or both even, $\alpha\mu_1\beta$ must be an odd permutation for any $(\alpha)(\beta) \in E(A!)$. But also we have $\rho_1\mu_1\rho_2 = \mu_2$ and

$\mu_2\mu_1\mu_2 = \mu_3$, so the class containing μ_1 is the entire set $\{\mu_1, \mu_2, \mu_3\}$ of odd permutations. It follows that the equivalence classes of $\simeq$ in this case are {id} and $\{\mu_1, \mu_2, \mu_3\}$. As was noted above, $A^{\mathrm{id}} \ncong A^{\mu_1} \cong A^{\mu_2} \cong A^{\mu_3}$. This illustrates a general principle.

Proposition 8. *If* $\lambda, \mu \in \mathrm{Ant}(A)$, *then* $\lambda \simeq \mu$ *if and only if* $A^\lambda \cong A^\mu$.

Proof. Suppose $\mu \simeq \lambda$, so $\mu = \alpha\lambda\beta$ for some $(\alpha)(\beta) \in E(A!)$. Then $\mu\beta^{-1} = \alpha\lambda$ and

$$\begin{aligned} aa' \in E(A) &\iff \alpha(a)\beta^{-1}(a') \in E(A) \\ &\iff \alpha(a)\mu\beta^{-1}(a') \in E(A^\mu) \\ &\iff \alpha(a)\alpha\lambda(a') \in E(A^\mu). \end{aligned}$$

Now, the edges of A^λ are precisely the pairs $a\lambda(a')$ for $aa' \in E(A)$, and the above equivalences show that $\alpha(a)\alpha(\lambda(a')) \in E(A^\mu)$. Thus α is a homomorphism from A^λ to A^μ. Further, observe that any edge $a\mu(a')$ of A^μ is the image under α of some edge of A^λ: Since $a\mu(a') \in A^\mu$, we have $aa' \in E(A)$, so $\alpha^{-1}(a)\beta(a') \in E(A)$, and hence $\alpha^{-1}(a)\lambda\beta(a') \in E(A^\lambda)$. Then α sends this edge to $a\,\alpha\lambda\beta(a') = a\mu(a')$. Therefore $\alpha : A^\lambda \to A^\mu$ is an isomorphism.

Conversely, let there be an isomorphism $\alpha : A^\lambda \to A^\mu$. Then $\mu = \alpha\lambda\lambda^{-1}\alpha^{-1}\mu = (\alpha)\lambda(\lambda^{-1}\alpha^{-1}\mu)$. We just need to show that $(\alpha)(\lambda^{-1}\alpha^{-1}\mu) \in E(A!)$, and this involves showing that $aa' \in E(A)$ if and only if $\alpha(a)\mu^{-1}\alpha\lambda(a') \in E(A)$. Now,

$$\begin{aligned} aa' \in E(A) &\iff a\lambda(a') \in E(A^\lambda) \\ &\iff \alpha(a)\alpha\lambda(a') \in E(A^\mu) \\ &\iff \alpha(a)\mu^{-1}\alpha\lambda(a') \in E(A) \quad \text{or } \mu^{-1}\alpha(a)\alpha\lambda(a') \in E(A). \end{aligned}$$

But if $\mu^{-1}\alpha(a)\alpha\lambda(a') \in E(A)$, the anti-automorphism property of μ implies that $\alpha(a)\mu^{-1}\alpha\lambda(a') \in E(A)$. ■

For each $\mu \in \mathrm{Ant}(A)$, let $[\mu]$ denote the $\simeq$ equivalence class containing μ. Propositions 5 and 8 imply the following.

Theorem 9. *Let A be a graph and C be a bipartite graph with at least one edge. If the equivalence classes of* $\mathrm{Ant}(A)$ *are* $\{[\mu_1], [\mu_2], \ldots, [\mu_k]\}$, *then the isomorphism classes of the graphs B for which* $A \times C \cong B \times C$ *are precisely those in* $\{A^{\mu_1}, A^{\mu_2}, \ldots, A^{\mu_k}\}$.

Let us call A a **cancellation graph** if $A \times C \cong B \times C$ implies $A \cong B$ for all graphs B and C (where C has at least one edge). Theorem 9 implies that A is a cancellation graph if and only if $\mathrm{Ant}(A)$ has only one $\simeq$ equivalence class. This leads to the following.

Theorem 10. *A graph A is a cancellation graph if and only if every anti-automorphism* μ *of A can be factored as* $\mu = \alpha\beta$ *where* $(\alpha)(\beta) \in E(A!)$.

Proof. Suppose A is a cancellation graph. Take $\mu \in \mathrm{Ant}(A)$. By Proposition 5, we have $A \times K_2 \cong A^\mu \times K_2$. But then the fact that A is a cancellation graph means $A \cong A^\mu$, which is to say $A^{\mathrm{id}} \cong A^\mu$. By Proposition 8 we have $\mu \simeq \mathrm{id}$ which means $\mu = \alpha\,\mathrm{id}\,\beta = \alpha\beta$ for some $(\alpha)(\beta) \in E(A!)$.

Conversely, suppose every $\mu \in \mathrm{Ant}(A)$ factors as $\mu = \alpha\beta$ for some $(\alpha)(\beta) \in E(A!)$. Suppose $A \times C \cong B \times C$. If C has an odd cycle, then $A \cong B$ by Theorem 3. If C is bipartite, then $B \cong A^\mu$ for some $\mu \in \mathrm{Ant}(A)$, by Proposition 5. Our assumption about μ implies $\mu \simeq \mathrm{id}$, so $A^\mu \cong A$. Thus $A \cong B$. ■

These results lead to some simple sufficient conditions for a graph to be a cancellation graph. For instance, A is a cancellation graph if $|\mathrm{Ant}(A)| = 1$. More generally, we have the following.

Corollary 11. *If every anti-automorphism of A has odd order, then A is a cancellation graph.*

Proof. Let μ be an anti-automorphism. Since $(\mu)(\mu) \in E(A!)$, the equation $\mu^3 = \mu\mu\mu$ gives $\mu^3 \simeq \mu$, and by iteration $\mu^p \simeq \mu$ for any odd integer p. Then $\mu \simeq \mathrm{id}$ whenever μ has odd order. ■

Finally, we have the following characterization for bipartite graphs. Recall that an involution is an automorphism of order 2.

Corollary 12. *A bipartite graph is a cancellation graph if and only if none of its components admits an involution that interchanges partite sets.*

The proof is omitted, since Corollary 12 was the main result of [3]. As an illustration of the corollary, The graph A in Fig. 1(b) has an involution that reverses its partite sets (reflection across a vertical axis) and indeed A does not have the cancellation property since $A \times C \cong B \times C$ but $A \ncong B$.

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