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Note

Robust cycle bases do not exist for $K_{n,n}$ if $n \geq 8$

Richard H. Hammack [a,*], Paul C. Kainen [b]

[a] Virginia Commonwealth University, Richmond, VA, USA
[b] Georgetown University, Washington DC, USA

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ABSTRACT

A basis for the cycle space of a graph is said to be *robust* if any cycle Z of G is a sum $Z = C_1 + C_2 + \cdots + C_k$ of basis elements such that (i) $(C_1 + C_2 + \cdots + C_{\ell-1}) \cap C_\ell$ is a nontrivial path for each $2 \leq \ell < k$. Hence, (ii) each partial sum $C_1 + C_2 + \cdots + C_\ell$ is a cycle for $1 \leq \ell \leq k$. While complete graphs and 2-connected plane graphs have robust cycle bases, it is shown that regular complete bipartite graphs $K_{n,n}$ do not have any robust cycle basis if $n \geq 8$.

1. Introduction

The problem of whether or not it is possible to find a graph with no robust basis has been open for nearly 20 years. We show that regular complete bipartite graphs $K_{n,n}$ have no robust bases when $n \geq 8$.

In the last five decades, cycle bases of graphs have been considered from novel perspectives. For instance, the minimum cycle basis problem asks for a cycle basis of smallest total length. Gleiss's dissertation [4] attributes this problem to Stepanec [17], and Zykov [20] in the Russian literature. M. Plotkin [16], a chemist, defined a graph cycle as *relevant* if it is not a sum of shorter cycles. Vismara [19] showed that a cycle is relevant if and only if it belongs to some minimum cycle basis.

Different questions were raised by Dixon and Goodman [2]. Their article seems to be the first appearance (in print) of the concept of a *weakly* robust basis, which is a cycle basis satisfying only the second condition (ii) given in the abstract. They conjectured that the bases associated with spanning trees are weakly robust. However, Sysło [18] gave a counter-example. Twenty years later, Dogrusöz and Krishnamoorthy [3] argued that for a 2-connected plane graph, the Mac Lane basis (the set of boundary cycles of the bounded regions) is weakly robust. Also, Ostermeier et al. [15] showed that the set of C_4-subgraphs containing a given edge of $K_{m,n}$ is weakly robust, and they gave a short proof of weak robustness for the Mac Lane basis.

The notion of a robust basis was formulated in [10], and applied to commutativity of diagrams. Also [10] proves that a robust basis of the complete graph K_n can be formed by taking all K_3-subgraphs containing a given vertex, and it notes that Mac Lane's basis of a 2-connected plane graph is robust. An explicit proof is given in [12], which further shows that no repeated terms are needed in the robust sums.

A substantial literature on cycle bases has developed (see, e.g., [6–9,14]). Applications have included the analysis of random protein networks [13], energy models for RNA folding [5] and commutativity of algebraic diagrams [11].

The remainder of the paper is organized as follows. In Section 2, we review the relevant background; results are proved in Section 3. The last section is a discussion.

* Corresponding author.
E-mail addresses: rhammack@vcu.edu (R.H. Hammack), kainen@georgetown.edu (P.C. Kainen).

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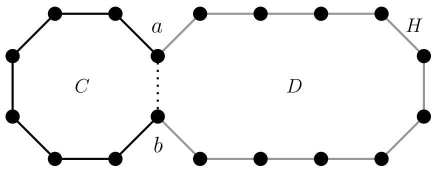

Fig. 1. A cycle C that is contiguous with a Hamiltonian cycle H.

2. Definitions

The **cycle space** $\mathscr{C}(G)$ of a graph G is the subset of the power set of $E(G)$ consisting of the subsets whose edge-induced subgraphs of G have no vertices of odd degree, endowed with the structure of a vector space over the two-element field $\mathbb{F}_2 = \{0, 1\}$. Addition is symmetric difference, and $\emptyset$ is the zero vector. Informally, one views $\mathscr{C}(G)$ as the set of spanning even-degree subgraphs of G, where the edgeless subgraph is zero. If G has c components, then $\mathscr{C}(G)$ has dimension $|E(G)| - |V(G)| + c$; see, e.g., Diestel [1, pp. 23–28]. A **cycle** in G is a 2-regular connected subgraph of G. Because any even-degree subgraph of G is the sum (possibly a trivial sum) of edge-disjoint cycles, $\mathscr{C}(G)$ is spanned by the cycles in G, and so has a basis whose elements are cycles. Such a basis for $\mathscr{C}(G)$ is called a **cycle basis**.

A cycle basis $\mathscr{B}$ of $\mathscr{C}(G)$ is a **weakly robust basis** if for each cycle Z in G there is a sequence $C_1, C_2, \ldots, C_k$ of elements of $\mathscr{B}$ (possibly with repetition) for which

$$Z = C_1 + C_2 + \cdots + C_k$$

and each partial sum $C_1 + C_2 + \cdots + C_\ell$ is a cycle for $1 \le \ell \le k$. The basis is called a **robust basis** if $(C_1 + C_2 + \cdots + C_\ell) \cap C_{\ell+1}$ is a nontrivial path for $1 \le \ell \le k - 1$. Here each summand is attached to the previous sum in a 1-cell, like a hinge. In such a case $C_1 + C_2 + \cdots + C_k$ is called a **robust sum**. In a robust basis, cycles are built by a sequence of attachments. Note that a robust basis is weakly robust. Acyclic graphs have empty bases, which are vacuously robust.

To see the difference between robust and weakly robust cycle bases, let G be a Möbius ladder. Embed G on a Möbius strip, and view the strip as a Möbius cap of the projective plane. Let $\mathscr{B}$ be the set of squares on the ladder, union a cycle in G with non-trivial homotopy in the projective plane. Check that $\mathscr{B}$ is a weakly robust basis but not a robust basis. (The boundary cycle of the ladder, which is homotopically trivial on the projective plane, is not a robust sum of basis elements.)

It is not known whether some graph has no weakly robust basis. Indeed, for bipartite complete graphs, such a weakly robust basis does exist. Ostermeier et al. [15] showed that the basis defined below satisfies weak robustness.

As in [10], we construct a cycle basis for $K_{n,m}$ as follows. Fix an edge ab. For any edge xy vertex-disjoint from ab, let S_{xy} denote the $K_{2,2} = C_4$-subgraph "*square*" induced by the set $\{a, b, x, y\}$ in $K_{n,m}$. The set of squares which contain ab,

$$\mathscr{K} := \mathscr{K}_{ab} := \{S_{xy} \mid xy \in E\left(K_{m,n} - \{a, b\}\right)\},$$

is independent because each square S_{xy} in $\mathscr{K}$ has the edge xy that belongs to no other, and so is a basis, as $|\mathscr{K}| = (m-1)(n-1) = mn - (m+n) + 1$ is the dimension of the cycle space of $K_{m,n}$. The basis $\mathscr{K}$ is called the **Kainen basis** in [9,14,15]. It is shown in [15] that $\mathscr{K}$ is robust if $m \le 4$ and $n \le 5$, and that $\mathscr{K}$ is not robust if $m, n \ge 5$. We will shortly prove the following slightly stronger result. (See also the Discussion section below.)

Proposition 1. *The basis* $\mathscr{K}$ *is robust if and only if* $\min\{m, n\} \le 4$.

This begs the question of whether or not *any* robust basis exists for $K_{m,n}$ when $\min\{m, n\} > 4$. We show the answer is "No" for $K_{n,n}$ with $n \ge 8$.

3. Results

This section depends on the following definition and proposition.

Definition 1. A cycle C in a graph G is **contiguous with** a Hamiltonian cycle H in G if at most one edge of C is not an edge of H. Thus C being contiguous with H means that either $H = C$, or $H = D + C$, where D is a cycle intersecting C precisely at an edge ab. (See in Fig. 1.)

In [12], two cycles are called *compatible* if they intersect in a nontrivial path. The nonidentity case of contiguity constrains the compatibility in two ways: $C + D$ must be spanning while $C \cap D$ is a path of one edge.

Proposition 2. *If* $\mathscr{B}$ *is a robust cycle basis for a graph* G*, and* H *is a Hamiltonian cycle in* G*, then there is some* $C \in \mathscr{B}$ *that is contiguous with* H.

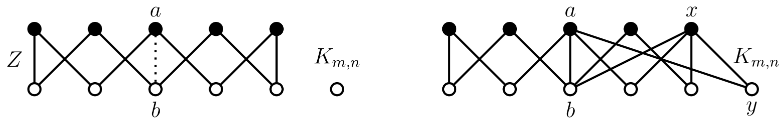

Fig. 2. Why the Kainen basis for $K_{m,n}$ is not robust when min $m, n > 4$.

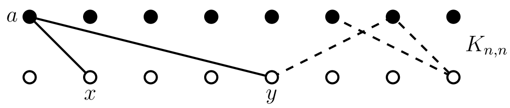

Fig. 3. Counting Hamiltonian cycles in $K_{n,n}$.

Proof. Let $\mathscr{B}$ be a robust cycle basis for G and let H be a Hamilton cycle in G. Then either $H \in \mathscr{B}$ (and we are done), or $H = C_1 + C_2 + C_3 + \cdots + C_k$, with each summand in $\mathscr{B}$, and where $(C_1 + C_2 + C_3 + \cdots + C_{\ell-1}) \cap C_\ell$ is a non-trivial path for each $1 < \ell \leq k$. Let $D = C_1 + C_2 + C_3 + \cdots + C_{k-1}$. Then $H = D + C_k$ and $D \cap C_k$ is a non-trivial path. But this path cannot have any internal vertices, for then they would not appear on the Hamiltonian cycle H. Thus $D \cap C_k$ is an edge, so C_k is contiguous with H. □

In what follows, we regard the vertices in one partite set of $K_{n,m}$ as colored black, and those in the other as colored white. We now prove Proposition 1, that the Kainen basis $\mathscr{K}$ of $K_{n,m}$ is robust if and only if $\min\{m, n\} \leq 4$.

Proof of Proposition 1. The statement is vacuously true if $\min\{m, n\} = 1$, so assume $2 \leq \min\{m, n\}$. For arbitrary m, n, the longest cycle in $K_{m,n}$ has length $2 \cdot \min\{m, n\}$. Thus it suffices to show that any cycle Z of length at most 8 in any $K_{m,n}$ is a robust sum of elements from $\mathscr{K}$.

First suppose that $Z = x_0x_1x_2 \cdots x_{2k-1}$ passes through neither a nor b, and without loss of generality, say that a and x_0 are in opposite partite sets. Note that $Z = S_{x_0x_1} + S_{x_1x_2} + S_{x_2x_3} + \cdots + S_{x_{2k-1}x_0}$ is a robust sum because $\left(S_{x_0x_1} + S_{x_1x_2} + \cdots + S_{x_{\ell-1}x_\ell}\right) \cap S_{x_\ell x_{\ell+1}} = P$, where P is the path abx_ℓ if ℓ is odd and $\ell < 2k - 1$, while $P = ax_\ell$ if ℓ is even. And finally, if $\ell = 2k - 1$, then P is the path $P = x_0abx_{2k-1}$.

Next, suppose Z passes through both a and b. If ab happens to be an edge of Z, then take a path $xaby$ in Z and note that $Z + S_{xy}$ is a robust sum equaling a cycle Z' that misses both a and b. Then $Z = Z' + S_{xy}$, and we can proceed by decomposing Z' as in the previous paragraph.

On the other hand, if Z passes through both a and b, but ab is not an edge of Z, then because the even cycle Z has length no greater than 8, it contains a path $axyb$. Then $Z + S_{xy} = Z'$ is a robust sum where Z' contains the edge ab. Then $Z = Z' + S_{xy}$, and we decompose Z' as in the previous paragraph.

Finally, suppose Z contains only one of a or b (say a). Take a path axy on Z. Notice $Z + S_{xy} = Z'$ is a robust sum and Z' is a cycle containing the edge ab. Then $Z = Z' + S_{xy}$, and we decompose Z' as before.

To see that $\mathscr{K}$ is not robust when $\min\{m, n\} > 4$, let Z be a cycle of length 10 in $K_{m,n}$, for which a and b are at distance 5 from each other in Z. (As shown on the left in Fig. 2.) Suppose to the contrary that $\mathscr{K}$ is robust.

If $m = n = 5$, then Z is Hamiltonian. Notice that in this case no element of $\mathscr{K}$ is contiguous with Z, contradicting Proposition 2. In general, for $n \geq m \geq 5$, the cycle Z is a robust sum

$$Z = S_{x_1,y_1} + \cdots + S_{x_\ell,y_\ell} + S_{x,y} \tag{1}$$

whose last term is some basis element $S_{x,y}$. The right of Fig. 2 shows the penultimate partial sum $S_{x_1,y_1} + \cdots + S_{x_\ell,y_\ell}$ for a typical final summand $S_{x,y}$. Observe that no matter the edge xy, the partial sum $S_{x_1,y_1} + \cdots + S_{x_\ell,y_\ell}$ is not a cycle, so the sum (1) cannot be robust, contrary to assumption. □

Having seen that the Kainen basis is robust only for $\min\{m, n\} \leq 4$, we now prove that in fact there does not exist *any* robust basis for $K_{n,n}$ when $n \geq 8$. Our approach uses a counting argument, involving Hamiltonian cycles, based on a well-known lemma. (The corresponding result for directed Hamiltonian cycles appears in Sequence A010790 in the Online Encyclopedia of Integer Sequences, http://oeis.org/.) For completeness, we give a short proof.

Lemma 1. *The graph $K_{n,n}$ has $\frac{n}{2}\left((n-1)!\right)^2$ Hamiltonian cycles.*

Proof. Fix a black vertex a of $K_{n,n}$. We will build a Hamiltonian cycle H through a by first choosing two white vertices x and y to be H-neighbors of a. There are $\binom{n}{2}$ ways to make this choice. Continuing the cycle from a through y, there are $n-1$ choices for the black vertex after y, then $n-2$ choices for the next white one, then $n-2$ for a black, then $n-3$ for a white, then $n-3$ for a black, etc. (See Fig. 3.) Thus the number of Hamiltonian cycles in $K_{n,n}$ is $\binom{n}{2}(n-1)(n-2)^2(n-3)^2 \cdots 1^2 = \frac{n}{2}\left((n-1)!\right)^2$. □

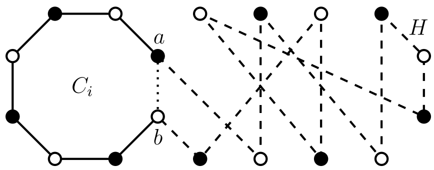

Fig. 4. Constructing Hamiltonian cycles that C_i is contiguous with.

Theorem 1. *If $K_{n,n}$ has a robust cycle basis, then $n \le 7$.*

Proof. Say $K_{n,n}$ has a robust cycle basis $\mathscr{B} = \{C_1, C_2, \ldots, C_p\}$, where $p = n^2 - 2n + 1 = (n-1)^2$, which is the dimension of the cycle space of $K_{n,n}$. As Proposition 1 asserts that such a robust basis exists when $n = 2, 3, 4$ we assume henceforward that $n \ge 4$.

In what follows we first show that $n \le 8$. Further analysis will then improve this to $n < 8$. We proceed via a sequence of claims.

Claim 1. *If $C_i \in \mathscr{B}$ has length $2k < 2n$, then it is contiguous with $2k\,((n-k)!)^2$ Hamiltonian cycles. And (obviously) if C_i has length $2k = 2n$ then it is contiguous with exactly one Hamiltonian cycle, namely itself.*

To prove this, take such a C_i of length $2k < 2n$. Select an edge ab of C_i, with a black and b white. Let us count the ways to extend $C_i - ab$ to a Hamiltonian cycle H. (That is, so that C_i is contiguous with H and ab is the only edge of C_i not on H.) We first run an edge from a to any of the $n-k$ white vertices in $V(H) - V(C_i)$. From that vertex, we may extend an edge to any of the $(n-k)$ black vertices in $V(H) - V(C_i)$. Then we extend to any of the $n-k-1$ remaining white vertices, then to any of the remaining $n-k-1$ black vertices, etc. (See Fig. 4.) In this way we see that C_i is contiguous with $((n-k)!)^2$ Hamiltonian cycles H in such a way that ab is the only edge of C_i not in H. As ab is one of $2k$ edges in C_i, it follows that C_i is contiguous with $2k((n-k)!)^2$ Hamiltonian cycles.

Claim 2. *If $3 \le k \le n$ and $4 \le n$, then $2k\,((n-k)!)^2 \le 2\,((n-2)!)^2$.*

For $k = 3$ the inequality holds by elementary algebraic inspection (using $n \ge 4$). Now assume $k > 3$. Notice that $2k \le 2\,((k-2)!)^2$ because beyond $k = 3$ the linear left-hand side is overtaken by the right-hand side. Using this with the fact $k \le n$, we get

$$\begin{aligned} 2k \le 2\,((k-2)!)^2 &= 2(k-2)^2(k-3)^2(k-4)^2\cdots(k-(k-1))^2 \\ &\le 2(n-2)^2(n-3)^2(n-4)^2\cdots(n-(k-1))^2 \\ &= \frac{2((n-2)!)^2}{((n-k)!)^2}. \end{aligned}$$

Comparing the first and last expressions yields $2k\,((n-k)!)^2 \le 2\,((n-2)!)^2$, confirming the claim.

Next we establish an upper bound on the number of Hamiltonian cycles in $K_{n,n}$. By Claim 1, with $k = 2$, any square in $\mathscr{B}$ is contiguous with $4\,((n-2)!)^2$ Hamiltonian cycles. Also, by Claim 1, if $C_i \in \mathscr{B}$ is not a square (that is, if it has length $2k$ with $k > 2$), then C_i is contiguous with $2k\,((n-k)!)^2$ Hamiltonian cycles, and, by Claim 2, this does not exceed $2\,((n-2)!)^2$. Conversely, Proposition 2 shows that each Hamiltonian cycle is counted since there is an element in $\mathscr{B}$ contiguous with it.

Let x be the number of elements of $\mathscr{B}$ that are squares; let y be the number of elements that are not squares (that is, have length greater than 4). By the above remarks, the total number of Hamiltonian cycles in $K_{n,n}$ does not exceed

$$x \cdot 4\,((n-2)!)^2 + y \cdot 2\,((n-2)!)^2\,.$$

Using Lemma 1,

$$\frac{n}{2}\,((n-1)!)^2 \le x \cdot 4\,((n-2)!)^2 + y \cdot 2\,((n-2)!)^2\,. \tag{2}$$

Thus $n(n-1)^2 \le 8x + 4y$. As $(n-1)^2 = |\mathscr{B}| = x + y$, we get $n(x+y) \le 8x + 4y$. Then

$$n \le 4 + \frac{4x}{x+y} = 4 + 4\frac{x}{|\mathscr{B}|}. \tag{3}$$

From this it follows that $n \le 8$. However, one more step improves the result to $n < 8$.

Claim 3. *Suppose that for any Hamiltonian cycle H of $K_{n,n}$, there is at most one square in $\mathscr{B}$ that is contiguous with H. Then $n = 4$. Otherwise $n \le 7$.*

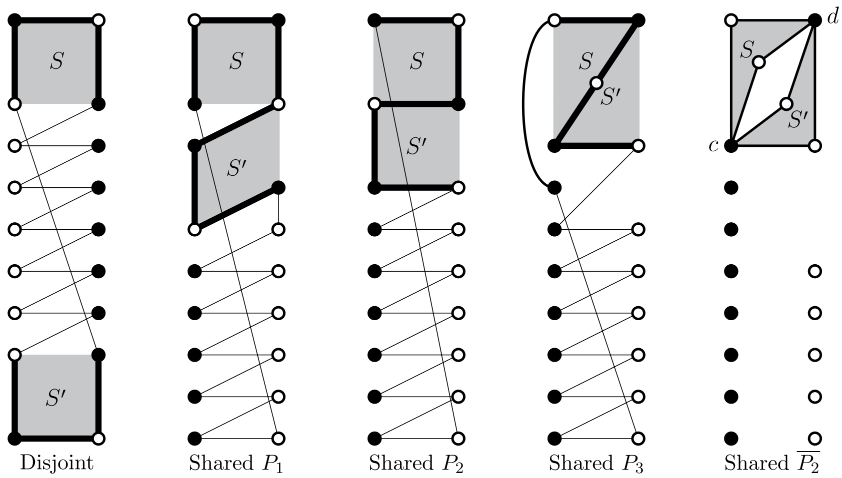

Fig. 5. The five ways that two squares on $K_{n,n}$ can intersect.

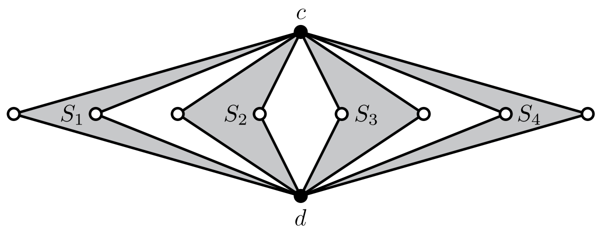

Fig. 6. The squares in $\mathscr{B}$.

To prove this, we count the ways two squares S, S' in $K_{n,n}$ can meet. Fig. 5 shows the five possibilities for the intersection: empty, a single vertex, a path of length 1, a path of length 2, or (in the last case) two nonadjacent vertices. The figure shows that in the first four cases S and S' are contiguous with a common Hamiltonian cycle.

But inspection reveals that in the last case (intersection at two vertices), S and S' are not contiguous with a common Hamiltonian cycle. By assumption, any two squares of $\mathscr{B}$ intersect in this way. Thus the squares in $\mathscr{B}$ are arranged as in Fig. 6, that is, any two of them intersect at a fixed set $\{c, d\}$ of vertices in the same partite set.

As each square uses two vertices of one of the partite sets, the number x of squares in $\mathscr{B}$ is no more than $\frac{n}{2}$. Note that for $n \geq 4$, we have $x \leq \frac{n}{2} < \frac{1}{4}(n-1)^2 = \frac{1}{4}|\mathscr{B}|$. Substituting this in inequality (3) yields $n \leq 4$, so $n = 4$.

Finally, if two squares in $\mathscr{B}$ are contiguous with the same Hamiltonian cycle, then we have double counted Hamiltonian cycles in the inequality (2), so it becomes strict, and the inequality (3) yields $n < 4 + 4 = 8$. □

4. Discussion

Proposition 1 says that $K_{n,n}$ has a robust basis for $n \leq 4$. By Proposition 1, no such basis exists for $n \geq 8$. The question is open for $n = 5, 6$, and 7.

The *robust span* of some family $\mathscr{F}$ of cycles in a graph G is the family $\rho(\mathscr{F})$ of all cycles with a robust sum from $\mathscr{F}$. Take $G = K_{n,n}$, with $n \geq 8$. For any basis $\mathscr{B}$ one has $\rho(\mathscr{B}) \subsetneq \mathrm{Cyc}(G)$, where $\mathrm{Cyc}(G)$ denotes the set of all cycle-subgraphs of G. However, the basis $\mathscr{K}$ will now be shown to be **iteratively robust** in that $\rho^k(\mathscr{K}) = \mathrm{Cyc}(G)$ for sufficiently large k, where the superscript on ρ means iterating the operation. Hence,

$$\rho(\mathscr{K}) \subsetneq \rho^2(\mathscr{K}). \tag{4}$$

To prove the iterative robustness of $\mathscr{K}$, recall that a cycle Z in G is **geodesic** if each pair of points in Z is joined by a G-geodesic path completely contained within Z. It is shown in [12, Thm. 6.1] that for any graph G, $\rho^k(\mathscr{G}) = \mathrm{Cyc}(G)$ for large enough k, where $\mathscr{G}$ denotes the family of all geodesic cycles in G. For $K_{n,n}$ a cycle is geodesic if and only if it has length 4. But the proof of Proposition 1 shows that each cycle of length at most 8 is a robust sum of cycles from $\mathscr{K}$ and hence $\mathscr{K}$ is iteratively robust.

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References

[1] R. Diestel, Graph Theory, third ed., in: Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005, p. xvi+411 ISBN:978-3-540-26182-7; 3-540-26182-6; 978-3-540-26183-4.
[2] E.T. Dixon, S.E. Goodman, An algorithm for the longest cycle problem, Networks 6 (2) (1976) 139–149.
[3] U. Dogrusoz, M.S. Krishnamoorthy, Enumerating all cycles of a planar graph, Parallel Algorithms Appl. (ISSN: 0028-3045) 10 (1–2) (1996) 21–36.
[4] P.M. Gleiss, Minimum Cycle Bases of Graphs from Chemistry (Ph.D. thesis), University of Vienna, 2001. https://www.tbi.univie.ac.at/papers/Abstracts/pmg_diss.pdf.
[5] P.M. Gleiss, P.F. Stadler, Relevant cycles in biopolymers and random graphs, in: Fourth Slovene International Conference in Graph Theory, 1999.
[6] R. Hammack, Minimum cycle bases of direct products of bipartite graphs, Australas. J. Combin. (ISSN: 1034-4942) 36 (2006) 213–221.
[7] M. Hellmuth, J. Leydold, P.F. Stadler, Convex cycle bases, Ars Math. Contemp. (ISSN: 1855-3966) 7 (1) (2014) 123–140.
[8] W. Imrich, P.F. Stadler, Minimum cycle bases of product graphs, Australas. J. Combin. (ISSN: 1034-4942) 26 (2002) 233–244.
[9] M.M.M. Jaradat, Minimal cycle bases of the lexicographic product of graphs, Discuss. Math. Graph Theory (ISSN: 1234-3099) 28 (2) (2008) 229–247. http://dx.doi.org/10.7151/dmgt.1403.
[10] P.C. Kainen, On robust cycle bases, in: The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications, in: Electron. Notes Discrete Math., vol. 11, Elsevier Sci. B. V., Amsterdam, 2002.
[11] P.C. Kainen, Graph cycles and diagram commutativity, Diagrammes (ISSN: 0224-3911) 67/68 (Liber Amicorum en l'honneur de Madame A.C. Ehresmann, suppl.) (2012) 179–237.
[12] P.C. Kainen, Cycle construction and geodesic cycles with application to the hypercube, Ars Math. Contemp. (ISSN: 1855-3966) 9 (1) (2015) 27–43.
[13] K. Klemm, P.F. Stadler, Statistics of cycles in large networks, Phys. Rev. E (3) 73 (025101) (2006).
[14] K. Klemm, P.F. Stadler, A note on fundamental, non-fundamental, and robust cycle bases, Discrete Appl. Math. (ISSN: 0166-218X) 157 (10) (2009) 2432–2438. http://dx.doi.org/10.1016/j.dam.2008.06.047.
[15] P. Ostermeier, M. Hellmuth, K. Klemm, J. Leydold, P.F. Stadler, A note on quasi-robust cycle bases, Ars Math. Contemp. (ISSN: 1855-3966) 2 (2) (2009) 231–240.
[16] M. Plotkin, Mathematical basis for ring finding algorithms in CIDS, J. Chem. Doc. 11 (1971) 60–63.
[17] G.F. Stepanec, Basis systems of vector cycles with extremal properties in graphs, Uspekhi Mat. Nauk (ISSN: 0042-1316) 19 (2 (116)) (1964) 171–175.
[18] M.M. Sysło, On cycle bases of a graph, Networks (ISSN: 0028-3045) 9 (2) (1979) 123–132. http://dx.doi.org/10.1002/net.3230090203.
[19] P. Vismara, Union of all the minimum cycle bases of a graph, Electron. J. Combin. (ISSN: 1077-8926) 4 (1) (1997) Research Paper 9, 15. http://www.combinatorics.org/Volume_4/Abstracts/v4i1r9.html.
[20] A.A. Zykov, Теория конечнюх графов I, Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk, 1969, p. 543.