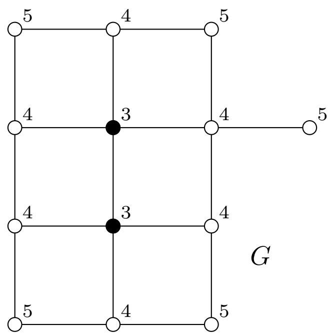
MATH 591: Graph Theory | Test | October 19, 2009

Name: ____________________ Score: ________

Directions: This is a closed-book, closed notes test. Please answer in the space provided. You *may not* use calculators, computers, etc.

1. (16 points) A graph G is drawn below. Label each vertex with its eccentricity. State the radius and diameter of G. Indicate the center of G.



The eccentricities are labeled above.
From this we can see that **the radius is 3**
and **the diameter is 5**.
The vertices in the center are darkened.

2. (6 points) Decide if the sequence $s: \; 4 \;\; 4 \;\; 4 \;\; 4 \;\; 3 \;\; 2$ is graphical. Show your work and/or explain your reasoning.

This sequence is not graphical: It has an odd number (1) of odd entries. If it were graphical, this graph would have and odd number of vertices of odd degree, and we know that is not possible.

3. (16 points) Suppose G is a graph of order n, and $\deg(v) \geq \frac{n-1}{2}$ for every $v \in V(G)$. Prove that G is connected.

We will prove the contrapositive: If G is not connected, then there is a vertex $v \in V(G)$ with $\deg(v) < \frac{n-1}{2}$. Thus suppose G is disconnected. Let X be the component of G with the fewest vertices. Then the number of vertices of X is at most half the number of vertices of G. Therefore $|V(X)| \leq \frac{|V(G)|}{2} = \frac{n}{2}$. Take any $v \in V(X)$. Since X has at most $n/2$ vertices, if follows that

$$\deg(v) \leq \frac{n}{2} - 1 = \frac{n-2}{2} < \frac{n-1}{2}.$$

The proof is now complete.

4. (10 points) Let G be a connected graph on at least three vertices, and let $e = uv$ be a bridge of G. Show that either u or v is a cut vertex of G.

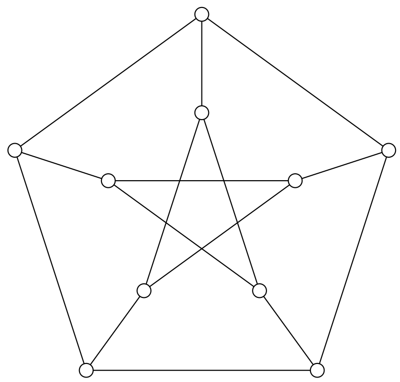
   Since e is a bridge, the graph $G - e$ is disconnected, and u and v are in different components of $G - e$. Since G has more than two vertices, there is a vertex $w \in V(G) - \{u, v\}$. Without loss of generality, we may assume w and u are in different components of $G - e$. Now think about $G - v$. This graph cannot have a path from u to w, because such a path would also be a path from u to w in $G - e$. Therefore $G - v$ is disconnected, so v is a cut vertex.

5. (12 points) What does it mean for a graph to be reconstructible? Give an example (with explanation) of a graph of small order that is not reconstructible.

   Given a graph G, the (multi)-set of its one-vertex-deleted subgraphs is the collection of all graphs $G - v$, for $v \in V(G)$. A graph G is reconstructible if whenever a graph H has the same one-vertex-deleted subgraphs as G, it is necessarily true that $G \cong H$.

   The graph $G = K_2$ is an example of a graph that is not reconstructible. The reason is that if we let $H = \overline{K_2}$, then the one-vertex-deleted subgraphs of G consist of two copies of K_1. But the one-vertex-deleted subgraphs of H are exactly the same, yet $G \ncong H$.

6. (12 points) Consider the Petersen Graph, sketched below.



Supply the following numeric information. (For *this* problem you do not have to justify your answers.)

(a) The connectivity is $\kappa(G) = 3$

(b) The edge-connectivity $\kappa_1(G) = 3$

(c) The toughness is $t(G) = \frac{3}{2}$

7. (6 points) Suppose a forest has 1000 vertices and 800 edges. How many components does it have?

For a forest F, we know that $|E(F)| = |V(F)| - k(F)$,
so therefore $800 = 1000 - k(F)$, and hence $k(F) = 200$.
The forest has 200 components.

8. (12 points) Suppose G is a connected planar graph with 16 vertices, each of degree 4. It is embedded in the plane so that every region is either a triangle or a quadrangle. How many triangles and how many quadrangles does this embedding have? Explain.

Since there are 16 vertices, each of degree 4, the number of edges is $|E(G)| = \frac{16 \cdot 4}{2} = 32$.
By Euler's law, $|V(G)| - |E(G)| + r = 2$, so the number of regions is $r = 2 + |E(G)| - |V(G)| = 2 + 32 - 16 = 18$.

Now let t be the number of triangular regions, and q be the number of quadrangle regions.
Since each edge is on the boundary of two regions, we get $3t + 4q = 2|E(G)| = 64$.

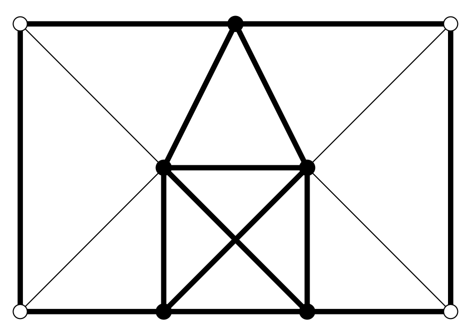
Now we have a system of two equations in two variables:

$$\begin{array}{rcrcl} t & + & q & = & 18 \\ 3t & + & 4q & = & 64 \end{array}$$

Eliminating t gives $q = 10$. Plugging this into the first equation gives $t = 8$.

ANSWER: The embedding has 8 triangles and 10 quadrangles.

9. (10 points) Establish the planarity or non-planarity of this graph. If it is planar, provide a rectilinear plane drawing.



A K_5 subdivison is shown with bold edges (and with vertices of K_5 darkened). Therefore, by Kuratowski's Theorem, **the graph is non-planar.**