CHAPTER 5

Exponential and Logarithmic Functions

We have examined power functions like $f(x) = x^2$. Interchanging x and 2 yields a different function $f(x) = 2^x$. This new function is radically different from a power function and has vastly different properties. It is called an *exponential function*. Exponential functions have many applications and play a big role in this course. Working with them requires understanding the basic laws of exponents. This chapter reviews these laws before recalling exponential functions. Then it explores inverses of exponential functions, which are called logarithms.

Recall that in an expression such as a^n in which a is raised to the power of n, the number a is called the **base** and n is the **exponent.**

5.1 Review of Exponents

We start at the beginning. For a number a and a positive integer n,

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}.$$

For example, $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$. This would be too elementary to mention except that every significant exponent property flows from it. For example,

$$(ab)^n = \underbrace{(ab) \cdot (ab) \cdot (ab) \cdots (ab)}_{n \text{ times}} = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}} \cdot \underbrace{b \cdot b \cdot b \cdots b}_{n \text{ times}} = a^n b^n,$$

that is, $(ab)^n = a^n b^n$. Also, $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} = \frac{a^n}{b^n}$. And $a^m a^n = a^{m+n}$ because

$$a^m a^n = \underbrace{a \cdot a \cdot a \cdots a}_{m \text{ times}} \cdot \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}} = a^{m+n}.$$

Assuming for the moment that $m > n$, we also get

$$\frac{a^m}{a^n} = \frac{\overbrace{a \cdot a \cdot a \cdot a \cdot a \cdots a}^{m \text{ times}}}{\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}} = \underbrace{a \cdot a \cdots \cdot a}_{m-n \text{ times}} = a^{m-n}$$

because the a's on the bottom cancel with n of the a's on top, leaving $m-n$ a's on top. Also notice that $(a^n)^m = a^{nm}$ because

$$(a^n)^m = \overbrace{\underbrace{(a \cdot a \cdot a \cdots a)}_{n \text{ times}} \cdot \underbrace{(a \cdot a \cdot a \cdots a)}_{n \text{ times}} \cdots \underbrace{(a \cdot a \cdot a \cdots a)}_{n \text{ times}}}^{m \text{ times}} = a^{nm}.$$

We have just verified the following fundamental Laws of exponents.

Basic Laws of Exponents

$$a^1 = a \qquad (ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^n)^m = a^{mn}$$

So far we have assumed n is a positive integer because in performing $a^n = a \cdot a \cdots a$ we cannot multiply a times itself a negative or fractional number of times. But there is a way to understand these rules when n is zero, negative or fractional. Trusting the above property $a^{m-n} = \frac{a^m}{a^n}$ yields

$$\begin{aligned} a^0 = a^{1-1} &= \frac{a^1}{a^1} = 1, \qquad (\text{provided } a \neq 0) \\ a^{-n} = a^{0-n} &= \frac{a^0}{a^n} = \frac{1}{a^n}. \end{aligned}$$

Notice 0^0 is undefined because above we have $0^0 = \frac{0^1}{0^1} = \frac{0}{0}$, which is undefined. But we can find a^n when n is 0 (and $a \neq 0$) or negative, as in $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$. In essence we just multiplied 2 times itself -3 times! Also note $a^{-1} = \frac{1}{a}$.

What about fractional powers, like $a^{1/n}$? If we believe $(a^n)^m = a^{nm}$, then

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \cdot n} = a^1 = a.$$

Thus $\left(a^{1/n}\right)^n = a$, meaning $a^{1/n} = \sqrt[n]{a}$. So $16^{1/4} = \sqrt[4]{16} = 2$ and $2^{1/2} = \sqrt{2}$. Further, $a^{\frac{m}{n}} = a^{\frac{1}{n}m} = \left(a^{\frac{1}{n}}\right)^m = \sqrt[n]{a}^{\,m}$. In summary:

$$a^0 = 1 \quad (\text{if } a \neq 0) \qquad a^{-n} = \frac{1}{a^n} \qquad a^{-1} = \frac{1}{a}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = \sqrt[n]{a}^{\,m} = \sqrt[n]{a^m}$$

The boxed equations hold for any rational values of m and n, positive or negative. We will use these frequently, and without further comment.

Example 5.1 Knowing the rules of exponents in the boxes above means we can evaluate many expressions without a calculator. For example, suppose we are confronted with $16^{-1.5}$. What number is this? We reckon

$$16^{-1.5} = 16^{-3/2} = \frac{1}{16^{3/2}} = \frac{1}{\sqrt{16}^3} = \frac{1}{4^3} = \frac{1}{64}.$$

For another example, $8^{-1.5} = 8^{-3/2} = \frac{1}{8^{3/2}} = \frac{1}{\sqrt{8}^3} = \frac{1}{(2\sqrt{2})^3} = \frac{1}{2^3\sqrt{2}^3} = \frac{1}{16\sqrt{2}}$.

Also, $(-8)^{5/3} = \sqrt[3]{-8}^5 = (-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$.

But if we attempt $(-8)^{5/2}$, we run into a problem because $(-8)^{5/2} = \sqrt{-8}^5$, and $\sqrt{-8}$ is not defined (or at least it is not a real number). In this case we simply say that $(-8)^{5/2}$ is not defined. ✍

It is important to be able to work problems such as these fluently, by hand or in your head. Over-reliance on calculators leads weak algebra skills, which will defeat you later in the course. If your algebra is rusty it is good practice to write everything out, without skipping steps. Algebra skills grow quickly through usage.

We have now seen how to evaluate a^p provided that p is a positive or negative integer or rational number (i.e., fraction of two integers). But not every power p falls into this category. For example, if $p = \pi$, then it is impossible to write $p = \frac{m}{n}$, as a fraction of two integers. How would we make sense of something like 2^π? One approach involves approximations. As $\pi \approx 3.14$, we have $2^\pi \approx 2^{3.14} = 2^{\frac{314}{100}} = \sqrt[100]{2}^{314}$. We could, at least in theory, arrive at very good approximations of 2^π with better approximations of π. We will drop this issue for now, because we will find a perfect way to understand a^x for *any* x once we have developed the necessary calculus. For now you should be content knowing that a^x has a value for any x, provided that a is a positive number.

Exercises for Section 5.1

Work the following exponents with pencil and paper alone. Then compare your answer to a calculator's to verify that the calculator is working properly.

1. $25^{1/2}$ **2.** $4^{1/2}$ **3.** $\frac{1}{4}^{1/2}$ **4.** $27^{1/3}$ **5.** $(-27)^{1/3}$ **6.** $(27)^{-1/3}$ **7.** $(-27)^{4/3}$

8. 2^{-1} **9.** 2^{-2} **10.** 2^{-3} **11.** $\frac{1}{2}^{-1}$ **12.** $\frac{1}{2}^{-2}$ **13.** $\frac{1}{2}^{-3}$ **14.** $\frac{1}{4}^{-1/2}$

15. $\sqrt{2}^6$ **16.** $\left(\frac{4}{9}\right)^{-1/2}$ **17.** $\left(\frac{\sqrt{3}}{2}\right)^{-4}$ **18.** $\frac{\sqrt{3}^{100}}{\sqrt{3}^{94}}$ **19.** $\left(\left(\frac{2}{3}\right)^{\frac{3}{2}}\right)^2$ **20.** $\left(\frac{3^9}{3^7}\right)^3$ **21.** $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$

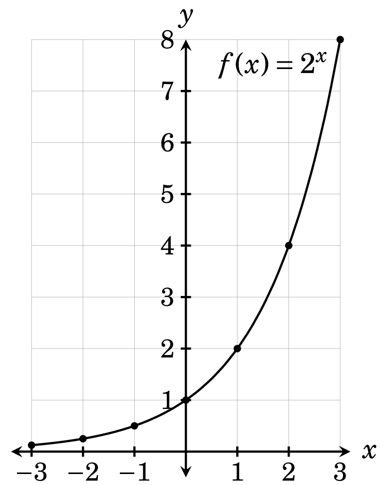
5.2 Exponential Functions

An **exponential function** is one of form $f(x) = a^x$, where a is a positive constant, called the **base** of the exponential function. For example $f(x) = 2^x$ and $f(x) = 3^x$ are exponential functions, as is $f(x) = \frac{1}{2}^x$.

If we let $a = 1$ in $f(x) = a^x$ we get $f(x) = 1^x = 1$, which is, in fact, a linear function. For this reason we agree that the base of an exponential function is never 1. To repeat, an exponential function has form $f(x) = a^x$, where a is a *positive* constant *unequal to 1*. We require a to be positive because we saw that in Example 5.1 that a^x may not be defined if a is negative.

Let's graph the exponential function $f(x) = 2^x$. Below is a table with some sample x and $f(x)$ values. The resulting graph is on the right.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

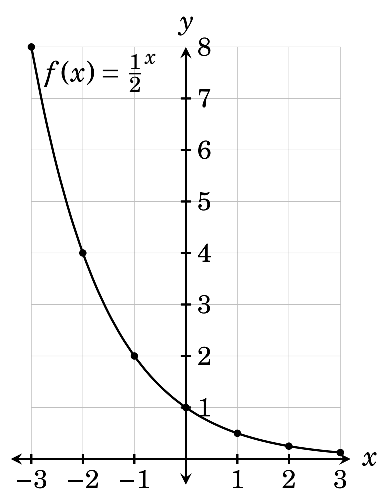


Notice that $f(x) = 2^x$ is positive for any x, but gets closer to zero the as x moves in the negative direction. But as $2^x > 0$ for any x, the graph never touches the x-axis.

The function $f(x) = 2^x$ grows very rapidly as x moves in the positive direction. Because $f(x) = 2^x$ is defined for all real numbers the domain of $f(x)$ is $\mathbb{R}$. The graph suggests that the range is all *positive* real numbers, that is, the range is the interval $(0, \infty)$.

Let's look at the exponential function $f(x) = \frac{1}{2}^x$. Here is its table and graph.

x	-3	-2	-1	0	1	2	3
$f(x) = \frac{1}{2}^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



Unlike $y = 2^x$, this function decreases as x increases. In fact, $f(x) = \frac{1}{2}^x = \frac{1^x}{2^x} = \frac{1}{2^x}$. As x increases towards infinity the denominator 2^x becomes ever bigger, so the fraction $f(x) = \frac{1}{2^x}$ gets closer and closer to zero. But no matter how big x gets, we still have $f(x) = \frac{1}{2^x} > 0$.

Figure 5.1 shows some additional exponential functions. It underscores the fact that the domain of any exponential function is $\mathbb{R} = (-\infty, \infty)$. The range is $(0, \infty)$. The y-intercept of any exponential function is 1.

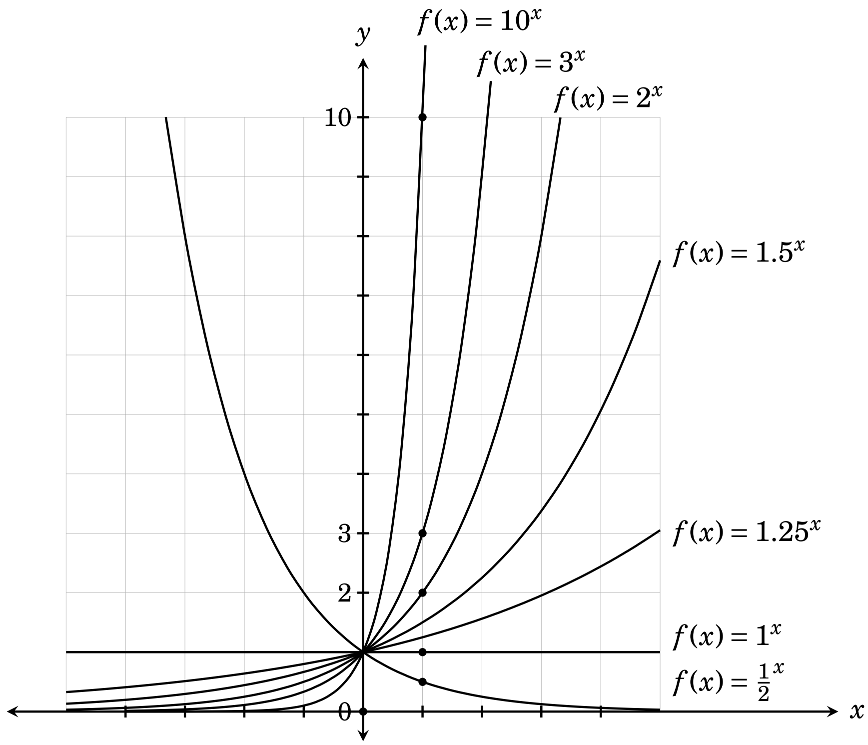


Figure 5.1. Some exponential functions. Each exponential function $f(x) = a^x$ has y-intercept $f(0) = a^0 = 1$. If $a > 1$, then $f(x) = a^x$ increases as x increases. If $0 < a < 1$, then $f(x) = a^x$ decreases as x increases.

Exercises for Section 5.2

1. Add the graphs of exponential functions $f(x) = \left(\frac{1}{10}\right)^x$ and $f(x) = \left(\frac{1}{3}\right)^x$ to Figure 5.1.

2. Add graphs of $f(x) = \left(\frac{1}{1.5}\right)^x = (0.\bar{6})^x$ and $f(x) = \left(\frac{1}{1.25}\right)^x = 0.8^x$ to Figure 5.1.

3. Use graph shifting techniques from Section 2.4 to draw the graph of $y = 2^{x-3} - 4$.

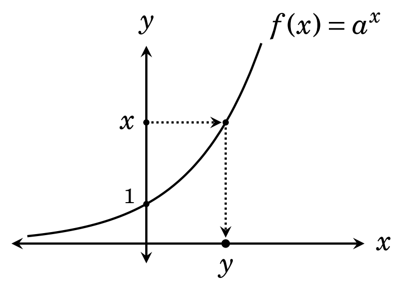
4. Use graph shifting techniques from Section 2.4 to draw the graph of $y = -2^{x+2}$.

5. Its graph shows that the exponential function $f(x) = 2^x$ is one-to-one. Use ideas from Section 4.2 to draw the graph of f^{-1}.

6. Consider the inverse of $f(x) = 2^x$. Find $f^{-1}(8)$, $f^{-1}(4)$, $f^{-1}(2)$, $f^{-1}(1)$, and $f^{-1}\left(\frac{1}{2}\right)$.

5.3 Logarithmic Functions

Now we apply the ideas of Chapter **??** to explore inverses of exponential functions. Such inverses are called *logarithmic functions*, or just *logarithms*.



An exponential function $f(x) = a^x$ is one-to-one and thus has an inverse. As illustrated above, this inverse sends any number x to the number y for which $f(y) = x$, that is, for which $a^y = x$. In other words,

$$f^{-1}(x) = \left(\begin{array}{c} \text{the number } y \\ \text{for which } a^y = x \end{array} \right).$$

From this it seems that a better name for f^{-1} might be $a^{\square}$, for then

$$a^{\square}(x) = \left(\begin{array}{c} \text{the number } y \\ \text{for which } a^y = x \end{array} \right).$$

The idea is that $a^{\square}(x)$ is the number y that goes in the box so that $a^y = x$. Using $a^{\square}$ as the name of f^{-1} thus puts the meaning of f^{-1} into its name. We therefore will use the symbol $a^{\square}$ instead of f^{-1} for the inverse of $f(x) = a^x$.

For example, the inverse of $f(x) = 2^x$ is a function called $2^{\square}$, where

$$2^{\square}(x) = \left(\begin{array}{c} \text{the number } y \\ \text{for which } 2^y = x \end{array} \right).$$

Consider $2^{\square}(8)$. Putting 3 in the box gives $2^3 = 8$, so $2^{\square}(8) = 3$. Similarly

$$\begin{array}{rl} 2^{\square}(16) = 4 & \quad \text{because } 2^4 = 16, \\ 2^{\square}(4) = 2 & \quad \text{because } 2^2 = 4, \\ 2^{\square}(2) = 1 & \quad \text{because } 2^1 = 2, \\ 2^{\square}(0.5) = -1 & \quad \text{because } 2^{-1} = \frac{1}{2} = 0.5. \end{array}$$

In the same spirit the inverse of $f(x) = 10^x$ is a function called $10^{\square}$, and

$$10^{\square}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } 10^y = x \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} 10^{\square}(1000) &= 3 && \text{because } 10^3 = 1000, \\ 10^{\square}(10) &= 1 && \text{because } 10^1 = 10, \\ 10^{\square}(0.1) &= -1 && \text{because } 10^{-1} = \tfrac{1}{10} = 0.1. \end{aligned}$$

Given a power 10^p of 10 we have $10^{\square}(10^p) = p$. For example,

$$\begin{aligned} 10^{\square}(100) &= 10^{\square}\left(10^2\right) = 2, \\ 10^{\square}\left(\sqrt{10}\right) &= 10^{\square}\left(10^{1/2}\right) = \tfrac{1}{2}. \end{aligned}$$

But doing, say, $10^{\square}(15)$ is not so easy because 15 is not an obvious power of 10. We will revisit this at the end of the section.

In general, the inverse of $f(x) = a^x$ is a function called $a^{\square}$, pronounced "*a box*," and defined as

$$a^{\square}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } a^y = x \end{pmatrix}.$$

You can always compute $a^{\square}$ of a power of a in your head because $a^{\square}(a^p) = p$.

The notation $a^{\square}$ is nice because it reminds us of the meaning of the function. But this book is probably the only place that you will ever see the symbol $a^{\square}$. Every other textbook—in fact all of the civilized world—uses the symbol $\log_a$ instead of $a^{\square}$, and calls it the *logarithm to base a*.

Definition 5.1 Suppose $a > 0$ and $a \neq 1$. The **logarithm to base a** is the function

$$\log_a(x) = a^{\square}(x) = \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } a^y = x \end{pmatrix}.$$

The function $\log_a$ is pronounced "*log base a*." It is the inverse of $f(x) = a^x$.

Here are some examples.

$$\begin{aligned} \log_2(8) &= 2^{\square}(8) = 3 & \log_5(125) &= 5^{\square}(125) = 3 \\ \log_2(4) &= 2^{\square}(4) = 2 & \log_5(25) &= 5^{\square}(25) = 2 \\ \log_2(2) &= 2^{\square}(2) = 1 & \log_5(5) &= 5^{\square}(5) = 1 \\ \log_2(1) &= 2^{\square}(1) = 0 & \log_5(1) &= 5^{\square}(1) = 0 \end{aligned}$$

To repeat, $\log_a$ and $a^{\square}$ are different names for the same function. We will bow to convention and use $\log_a$, mostly. But we will revert to $a^{\square}$ whenever it makes the discussion clearer.

Understanding the graphs of logarithm functions is important. Because $\log_a$ is the inverse of $f(x) = a^x$, its graph is the graph of $y = a^x$ reflected across the line $y = x$, as illustrated in Figure 5.2.

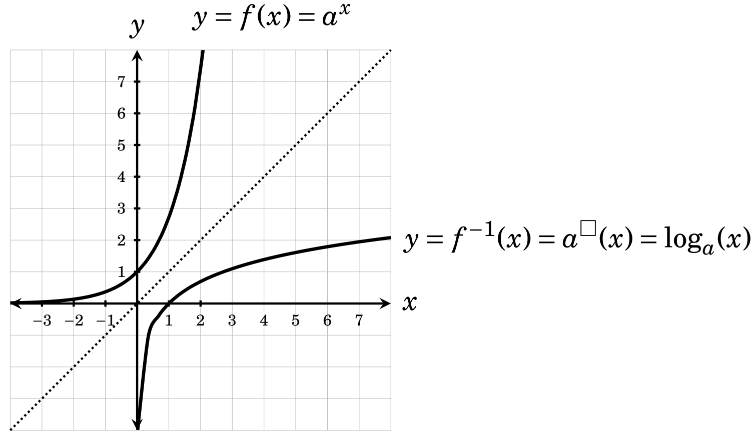


Figure 5.2. The exponential function $y = a^x$ and its inverse $y = \log_a(x)$. Notice $\log_a(x)$ is negative if $0 < x < 1$, Also $\log_a(x)$ tends to $-\infty$ as x gets closer to 0.

Take note that the domain of $\log_a$ is all positive numbers $(0,\infty)$ because this is the *range* of a^x. Likewise the range of $\log_a$ is the domain of a^x, which is $\mathbb{R}$. Also, because $\log_a(1) = a^{\square}(1) = 0$, the x-intercept of $y = \log_a(x)$ is 1.

The logarithm function $\log_{10}$ to base 10 occurs frequently enough that it is abbreviated as log and called the *common logarithm*.

Definition 5.2 The **common logarithm** is the function log defined as

$$\log(x) = \log_{10}(x) = 10^{\square}(x).$$

Most calculators have a $\boxed{\log}$ button for the common logarithm. Test this on your calculator by confirming that $\log(1000) = 3$ and $\log(0.1) = -1$. The button will also tell you that $\log(15) = 1.17609125\cdots$. In other words $10^{\square}(15) = 1.17609125\cdots$, which means $10^{1.17609125\cdots} = 15$, a fact with which your calculator will concur.

One final point. Convention allows for $\log_a x$ in the place of $\log_a(x)$, that is, the parentheses may be dropped. We will tend to use them.

Exercises for Section 5.3

1. $\log_3(81) =$ **2.** $\log_3\left(\frac{1}{9}\right) =$ **3.** $\log_3(\sqrt{3}) =$ **4.** $\log_3\left(\frac{1}{\sqrt{3}}\right) =$ **5.** $\log_3(1) =$

6. $\log(1000) =$ **7.** $\log(\sqrt[3]{10}) =$ **8.** $\log(\sqrt[3]{100}) =$ **9.** $\log(0.01) =$ **10.** $\log(1) =$

11. $\log_4(4) =$ **12.** $\log_4(2) =$ **13.** $\log_4(\sqrt{2}) =$ **14.** $\log_4(16) =$ **15.** $\log_4(8) =$

16. Simplify $\log_2(2^{\sin(x)})$. **17.** Simplify $10^{\log(5)}$.

18. Draw the graphs of $y = 10^x$ and its inverse $y = \log(x)$.

19. Draw the graphs of $y = 3^x$ and its inverse $y = \log_3(x)$.

5.4 Logarithm Laws

The function $\log_a$, also called $a^{\square}$, is the inverse of the exponential function $f(x) = a^x$. Consequently (as this section will show) the laws of exponents outlined in Section 5.1 come through as corresponding laws of $\log_a$.

To start, for any x it is obvious that $a^{\square}(a^x) = x$ because x is what must go into the box so that a to that power equals a^x. So we have

$$\begin{aligned} a^{\square}(a^x) &= x, \\ \log_a(a^x) &= x. \end{aligned} \tag{5.1}$$

This simply reflects the fact that $f^{-1}\big(f(x)\big) = x$ for the function $f(x) = a^x$.

Next consider the expression $a^{a^{\square}(x)}$. Here a is being raised to the power $a^{\square}(x)$, which is literally the power a must be raised to to give x. Therefore

$$\begin{aligned} a^{a^{\square}(x)} &= x, \\ a^{\log_b(x)} &= x \end{aligned} \tag{5.2}$$

for any x in the domain of $a^{\square}$. This is just saying $f\big(f^{-1}(x)\big) = x$.

The x in Equations (5.1) and (5.2) can be any appropriate quantity or expression. It is reasonable to think of these equations as saying

$$a^{\square}\left(a^{\blacksquare}\right) = \blacksquare \quad \text{and} \quad a^{a^{\square}(\blacksquare)} = \blacksquare,$$

where the gray rectangle can represent an arbitrary expression. For examples, $a^{\square}\left(a^{x+y^2+3}\right) = x + y^2 + 3$ and $a^{a^{\square}\left(\sin(\theta)\right)} = \sin(\theta)$.

Now we will verify a very fundamental formula for $\log_a(xy)$. Notice

$$
\begin{aligned}
a^{\square}(xy) &= a^{\square}\left(a^{a^{\square}(x)}a^{a^{\square}(y)}\right) && \text{(using } x = a^{a^{\square}(x)} \text{ and } y = a^{a^{\square}(y)}\text{)}\\
&= a^{\square}\left(a^{a^{\square}(x)+a^{\square}(y)}\right) && \text{(because } a^c a^d = a^{c+d}\text{)}\\
&= a^{\square}(x) + a^{\square}(y) && \text{(using } a^{\square}\left(a^{\blacksquare}\right) = \blacksquare\text{)}
\end{aligned}
$$

We have therefore established

$$
\begin{aligned}
a^{\square}(xy) &= a^{\square}(x) + a^{\square}(y),\\
\log_a(xy) &= \log_a(x) + \log_a(y).
\end{aligned}
\tag{5.3}
$$

By the same reasoning you can also show $a^{\square}\left(\frac{x}{y}\right) = a^{\square}(x) - a^{\square}(y)$, that is,

$$
\begin{aligned}
a^{\square}\left(\frac{x}{y}\right) &= a^{\square}(x) - a^{\square}(y),\\
\log_a\left(\frac{x}{y}\right) &= \log_a(x) - \log_a(y).
\end{aligned}
\tag{5.4}
$$

Applying $a^{\square}(1) = 0$ to this yields $a^{\square}\left(\frac{1}{y}\right) = a^{\square}(1) - a^{\square}(y) = -a^{\square}(y)$, so

$$
\begin{aligned}
a^{\square}\left(\frac{1}{y}\right) &= -a^{\square}(y),\\
\log_a\left(\frac{1}{y}\right) &= -\log_a(y).
\end{aligned}
\tag{5.5}
$$

Here is a summary of what we have established so far.

Logarithm Laws

$\log_a(a^x) = x$	$\log_a(1) = 0$	$\log_a(xy) = \log_a(x) + \log_a(y)$	$\log_a(x^y) = y\log_a(x)$
$a^{\log_a(x)} = x$	$\log_a(a) = 1$	$\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$	$\log_a\left(\dfrac{1}{y}\right) = -\log_a(y)$

The one law in this list that we have not yet verified is $\log_a(x^y) = y\log_a(x)$. This is an especially useful rule because it says that taking $\log_a$ of x^y converts the *exponent* y to a *product*. Because products tend to be simpler than exponents, this property is tremendously useful in many situations. To verify it, just notice that

$$
\begin{aligned}
a^{\square}(x^y) &= a^{\square}\left(\left(a^{a^{\square}(x)}\right)^y\right) && \text{(because } x = a^{a^{\square}(x)}\text{)}\\
&= a^{\square}\left(a^{y a^{\square}(x)}\right) && \text{(because } \left(a^b\right)^y = a^{yb}\text{)}\\
&= y a^{\square}(x) && \text{(using } a^{\square}\left(a^{\blacksquare}\right) = \blacksquare\text{)}
\end{aligned}
$$

Therefore $a^{\square}(x^y) = y\,a^{\square}(x)$, or $\log_a(x^y) = y\log_a(x)$, as listed above.

By the above laws, certain expressions involving logarithms can be transformed into simpler expressions.

Example 5.2 Write $\log_2(5) + \frac{1}{2}\log_2(x+1) - \log_2(5x)$ as an expression with a single logarithm.

To solve this we use the laws of logarithms, and work as follows.

$$\begin{aligned}
\log_2(5) + \frac{1}{2}\log_2(x+1) - \log_2(5x) &= \log_2(5) + \log_2\left((x+1)^{1/2}\right) - \log_2(5x) \\
&= \log_2(5) + \log_2\left(\sqrt{x+1}\right) - \log_2(5x) \\
&= \log_2\left(5\sqrt{x+1}\right) - \log_2(5x) \\
&= \log_2\left(\frac{5\sqrt{x+1}}{5x}\right) \\
&= \log_2\left(\frac{\sqrt{x+1}}{x}\right)
\end{aligned}$$

✍

Example 5.3 Break the expression $\log\sqrt{\dfrac{x\sin(x)}{10^x}}$ into simpler logarithms.

Recall that log is the common logarithm $\log_{10}$. Using the laws of logarithms,

$$\begin{aligned}
\log\sqrt{\frac{x\sin(x)}{10^x}} &= \log\left(\left(\frac{x\sin(x)}{10^x}\right)^{1/2}\right) \\
&= \frac{1}{2}\log\left(\frac{x\sin(x)}{10^x}\right) \\
&= \frac{1}{2}\left(\log(x\sin(x)) - \log(10^x)\right) \\
&= \frac{1}{2}\left(\log(x\sin(x)) - x\right) \\
&= \frac{1}{2}\left(\log(x) + \log(\sin(x)) - x\right) \\
&= \frac{\log(x)}{2} + \frac{\log\left(\sin(x)\right)}{2} - \frac{x}{2}.
\end{aligned}$$

The final answer is somewhat subjective. What do we mean by "simpler" logarithms? An alternate answer would be $\log(\sqrt{x}) + \log\left(\sqrt{\sin(x)}\right) - \frac{x}{2}$. ✍

Example 5.4 Simplify $\log_2(28x) - \log_2(7x)$.

To solve this we use the laws of logarithms to get

$$\begin{aligned}\log_2(28x) - \log_2(7x) &= \log_2\left(\frac{28x}{7x}\right)\\ &= \log_2(4)\\ &= 2.\end{aligned}$$

Now work a few practice problems. ✍

Exercises for Section 5.4

1. Write as a single logarithm: $5\log_2(x^3+1) + \log_2(x) - \log_2(3)$
2. Write as a single logarithm: $\log_2(\sin(x)) + \frac{1}{2}\log_2(4x) - 3\log_2(3)$
3. Write as a single logarithm: $2 + \log(5) + 2\log(7)$
4. Write as a single logarithm: $\log(2x) + \log(5x)$
5. Break into simpler logarithms: $\log_2\sqrt{\frac{x^3}{x+1}}$
6. Break into simpler logarithms: $\log_2\left((x+5)^4 x^7 \cos(x)\right)$
7. Break into simpler logarithms: $\log\left(\sqrt{x}(x+3)^6\right)$
8. Break into simpler logarithms: $\log_3\left(\frac{3}{5\sqrt[3]{x}}\right)$
9. Simplify: $\log\left(10x^{10}\right)$
10. Simplify: $\log(2) + \log(5)$
11. Simplify: $\log(2) + \log(2x) + \log(25x)$
12. Simplify: $\log_2(2) - \log_2(5x) + \log_2(20x)$
13. Find the inverse of the function $f(x) = 7 - 2^{x^3+4}$
14. Find the inverse of the function $f(x) = 4\log_2(x) + 3$
15. Find the inverse of the function $g(\theta) = \frac{2}{5^\theta} + 1$
16. Find the inverse of the function $g(z) = 2^{1/z}$
17. Use graph shifting to sketch the graph of $y = -\log_2(x) + 1$.
18. Use graph shifting to sketch the graph of $y = \log(x+4)$.

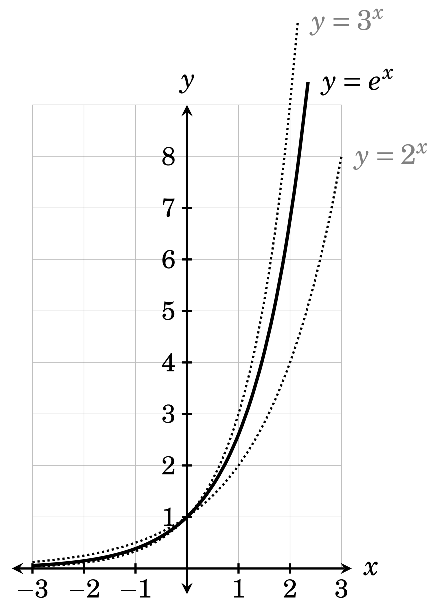
5.5 The Natural Exponential and Logarithm Functions

Calculus involves two very significant constants. One of them is π, which is essential to the trigonometric functions. The other is a mysterious number $e = 2.718281829459\cdots$ that arises from exponential functions. Like π, the number e is irrational – it is not a fraction of integers and its digits do not repeat. But unlike π (the ratio of circumference to diameter of a circle) it is difficult to justify the importance of e without first developing some calculus. This will come in due time. Eventually we will find formulas for e. For now we merely remark that e will turn out to be very significant. The exponential function e^x is called the *natural exponential function*.

The **natural exponential function** is the function $f(x) = e^x$.

The natural exponential function $y = e^x$ is graphed on the right. Because $2 < e < 3$, this graph lies between the graphs of the exponential functions $y = 2^x$ and $y = 3^x$, whose graphs are sketched in gray.

The natural exponential function shares features of other exponential functions. It grows very rapidly as x moves in the positive direction. Taking x in the negative direction, e^x becomes closer and closer to 0. Its domain of $f(x) = e^x$ is $\mathbb{R}$ and the range is $(0, \infty)$.



Later we will (for the most part) forsake all other exponential functions and treat the natural exponential function as if it is the only exponential function. There are good reasons for doing this, as we will see.

Section 5.3 tells us the inverse of $f(x) = e^x$ is $f^{-1}(x) = \log_e(x)$. This logarithm occurs so often that it is called the *natural logarithm* and is abbreviated as $\ln(x)$. Whenever you see $\ln(x)$ it means $\log_e(x)$.

Definition 5.3 The function $f(x) = e^x$ is called the **natural exponential function**. Its inverse is called the **natural logarithm** and is denoted as $\ln(x)$, that is, $\ln(x) = \log_e(x)$.

Figure 5.3 shows the graph of both e^x and $\ln(x)$. Because they are inverses of one another, the graph of one is the reflection of the other's graph across the line $y = x$. Take note that the positive real numbers constitute the domain of $\ln(x)$. Its range if $\mathbb{R}$.

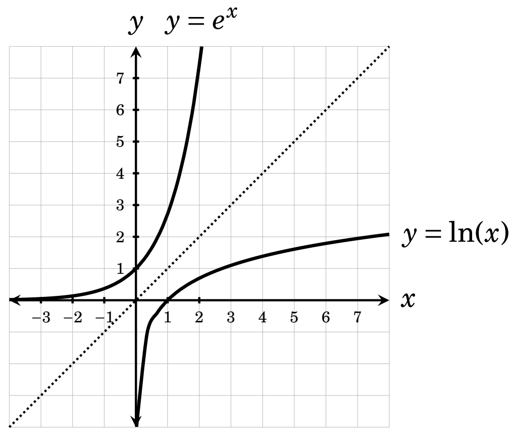


Figure 5.3. The natural exponential function e^x and its inverse $\ln(x)$

Never forget the meaning of ln. It is $\log_e$, which we also call $e^{\square}$. Its meaning is simply

$$\ln(x) = e^{\square}(x) = \begin{pmatrix} \text{power } y \text{ for} \\ \text{which } e^y = x \end{pmatrix}. \tag{5.6}$$

For example, $\ln(1) = e^{\square}(1) = 0$ because $e^0 = 1$. Also $\ln(e) = e^{\square}(e) = 1$ because putting 1 in the box gives e. In general $\ln(e^x) = e^{\square}(e^x) = x$, so

$$\begin{aligned}
\ln\left(\frac{1}{e}\right) &= \ln\left(e^{-1}\right) = e^{\square}(e^{-1}) = -1, \\
\ln(\sqrt{e}) &= \ln\left(e^{1/2}\right) = e^{\square}(e^{1/2}) = \tfrac{1}{2}, \\
\ln\left(\frac{1}{\sqrt{e}^3}\right) &= \ln\left(e^{-3/2}\right) = e^{\square}(e^{-3/2}) = -\tfrac{3}{2}, \\
\ln(e^x) &= e^{\square}(e^x) = x.
\end{aligned}$$

This last equation $\ln(e^x) = x$ is simply $f^{-1}\big(f(x)\big) = x$ for the case $f(x) = e^x$.

Of course we can't always work out $\ln(x)$ mentally, as was done above. If confronted with, say, $\ln(9)$ we might think of it as $e^{\square}(9)$ but we're stuck because no obvious number goes in the box to give 9. The best we can do is guess that because $e \approx 2.7$ is close to 3 and $3^2 = 9$, then $\ln(9)$ must be close to 2. Fortunately, a good calculator has a $\boxed{\ln}$ button. Using it, we find $\ln(9) \approx 2.1972$.

Although the $e^{\square}$ notation may be a helpful nemonic device, we will not use it extensively, and you will probably drop it yourself once you have become fluent in logarithms. (Perhaps that has already happened.)

Please be aware that it is common (particularly in other texts) to drop the parentheses and write $\ln(x)$ simply as $\ln x$. For clarity this text will mostly retain the parentheses, we will certainly drop them occasionally.

Because $\ln = \log_e$, all standard logarithm properties apply to it.

Natural Logarithm Laws

$\ln(e^x) = x$	$\ln(1) = 0$	$\ln(xy) = \ln(x) + \ln(y)$	$\ln(x^r) = r\ln(x)$
$e^{\ln(x)} = x$	$\ln(e) = 1$	$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$	$\ln\left(\frac{1}{y}\right) = -\ln(y)$

Example 5.5 Simplify $\ln(25) + 2\ln\left(\frac{e}{5}\right)$.

By natural log laws this is $\ln(5^2) + 2\big(\ln(e) - \ln(5)\big) = 2\ln(5) + 2\ln(e) - 2\ln(5) = 2\ln(e) = 2 \cdot 1 = 2$. ✍

The law $\ln(x^r) = r\ln(x)$ is extremely useful because it means taking ln of x^r converts the exponent r to a product. Consequently ln can be used to solve an equation for a quantity that appears as an exponent.

Example 5.6 Solve the equation $5^{x+7} = 2^x$.
In other words we want to find the value of x that makes this true. Since x occurs as an exponent, we take ln of both sides and simplify with log laws.

$$\begin{aligned}
\ln\left(5^{x+7}\right) &= \ln\left(2^x\right) \\
(x+7)\cdot\ln(5) &= x\cdot\ln(2) \\
x\ln(5) + 7\ln(5) &= x\ln(2) \\
x\ln(5) - x\ln(2) &= -7\ln(5) \\
x\big(\ln(5) - \ln(2)\big) &= -7\ln(5) \\
x &= \frac{-7\ln(5)}{\ln(5) - \ln(2)} \\
x &\approx -12.2952955815,
\end{aligned}$$

where we have used a calculator in the final step. ✍

In doing this problem we could have used a logarithm to *any* base, not just $\ln = \log_e$. Most calculators have a $\boxed{\text{log}}$ button for the common logarithm

$\log_{10}$. With log the steps above gives the same solution

$$x = \frac{-7\log(5)}{\log(5) - \log(2)} \approx -12.2952955815.$$

The next two examples will lead to a significant formula called the *change of base formula*.

Example 5.7 Suppose b is positive. Solve the equation $b^y = x$ for y.

The variable y is an exponent, so we take ln of both sides and simplify.

$$\begin{aligned} \ln\left(b^y\right) &= \ln(x) \\ y\ln(b) &= \ln(x) \\ y &= \frac{\ln(x)}{\ln(b)} \end{aligned}$$

Therefore, in terms of x and b, the quantity y is the number $\frac{\ln(x)}{\ln(b)}$. ✍

Example 5.7 would have been quicker if we had used $\log_b$ instead of ln. Lets's do the same problem again with this alternative approach.

Example 5.8 Suppose b is positive. Solve the equation $b^y = x$ for y.

The variable y is an exponent, so we take $\log_b$ of both sides and simplify.

$$\begin{aligned} \log_b\left(b^y\right) &= \log_b(x) \\ y &= \log_b(x) \end{aligned}$$

Therefore, in terms of x and b, the quanitty y is the number $\log_b(x)$. ✍

Examples 5.7 and 5.8 say the solution of $b^y = x$ can be expressed either as $\frac{\ln(x)}{\ln(b)}$ or $\log_b(x)$. The first solution may be preferable, as your calculator has no $\boxed{\log_b}$ button. But what is significant is that these two methods arrive at the *same* solution, which is to say $\log_b(x) = \frac{\ln(x)}{\ln(b)}$. To summarize:

> **Fact 5.1** (Change of Base Formula)
>
> $$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

The change of base formula says that a logarithm $\log_b(x)$ to *any* base b can be expressed entirely in terms of the natural logarithm ln.

Example 5.9 By the change of base formula, $\log_2(5) = \dfrac{\ln(5)}{\ln(2)} \approx \dfrac{1.6094379}{0.6931471} = 2.3219280$. This seems about right because $\log_2(5) = 2^{\square}(5)$ is the number y for which $2^y = 5$. Now, $2^2 = 4 < 5 < 8 = 2^3$, so y should be between 2 and 3. This example shows in fact $y = 2.3219280$ to seven decimal places. ✍

Exercises for Section 5.5

Simplify the following expressions.

1. $\ln\left(\dfrac{1}{e}\right)$ **2.** $2\ln(7e) - \ln(49)$ **3.** $e^{\sqrt{2}} \cdot e^{-\sqrt{2}}$

4. $2\ln(6) - \ln(4) - \ln(9)$ **5.** $e^{\ln(3)-\ln(2)}$ **6.** $e^{5\ln(3)}$

Use the techniques of this section to solve the following equations.

7. $2^{x+3} = 3^x$ **8.** $10^{4x-2} = 11$ **9.** $e^{4x-2} = 3e^2$ **10.** $e^{x^2-2+1} = 1$

11. $e^{\sin(x)} = e$ **12.** $2^{\sin(x)} = 2^{\cos(x)}$ **13.** $10^x = 17$ **14.** $\ln(x^2 + 5x + 7) = 0$

Use the change of base formula to express the following logarithms in terms of ln.

15. $\log_2(5)$ **16.** $\log_3(5)$ **17.** $\log_4(5)$ **18.** $\log_5(5)$

19. $\log(8)$ **20.** $\log(9)$ **21.** $\log(10)$ **22.** $\log(99)$

23. $\log_3(8)$ **24.** $\log_3(9)$ **25.** $\log_3(10)$ **26.** $\log_3(29)$

5.6 The Significance of e

What is this mysterious number $e \approx 2.71828182$ and why is it important? In this section we will first define e by giving a formula for it. Then we explore one reason why the number e is significant.

The number e comes from the value of the expression $\left(\frac{1}{n}+1\right)^n$, where n is a positive integer. Consider the table below. The left column lists integers $1,2,3,\ldots$, increasing in value and getting quite large. The right column gives the corresponding value of $\left(\frac{1}{n}+1\right)^n$. For example, in the first line we have $n=1$ and $\left(\frac{1}{n}+1\right)^n = \left(\frac{1}{1}+1\right)^1 = 2$. The second line has $n=2$ and $\left(\frac{1}{2}+1\right)^2 = 1.5^2 = 2.25$. The later lines—done with a calculator—are accurate to 15 decimal places if an exact value is not possible.

n	$\left(\dfrac{1}{n}+1\right)^n$
1	2
2	2.25
3	2.370270370370370
4	2.44140625
5	2.48832
10	2.5937424601
100	2.704813829421526
1000	2.716923932235892
10000	2.718145926825225
100000	2.718268237174495
1000000	2.718280469319337
10000000	2.718281692544966
100000000	2.718281814867636
$\downarrow$	$\downarrow$
∞	e

Notice how as n increases, the number $\left(\frac{1}{n}+1\right)^n$ increases too, but slowly. As n increases to ∞, the number $\left(\frac{1}{n}+1\right)^n$ appears to stabilize at some value around 2.718281828. This is e. The number e is the value that $\left(\frac{1}{n}+1\right)^n$ approaches as n gets bigger and bigger.

This is an example of what is called a *limit*, a topic that we will explore in Part 2 of this text. But for now it suffices to say that for very large values of n, the quantity $\left(\frac{1}{n}+1\right)^n$ is very close to e. For example, putting in $n = 10^{16}$ gives e to 15 decimal places:

$$e \approx 2.718281828459045\cdots.$$

If you want more accuracy, use a larger n.

So why is e significant? For us, the reason is this: We know that any exponential function $y = a^x$ has y-intercept $(0, a^0) = (0,1)$, and the graph of $y = a^x$ has a tangent line at this point $(0,1)$. The function $y = e^x$ is *the only* exponential function for which this tangent has slope 1. Figure 5.4 illustrates this. The line $y = x+1$ (with slope 1) is tangent to $y = e^x$ at $(0,1)$.

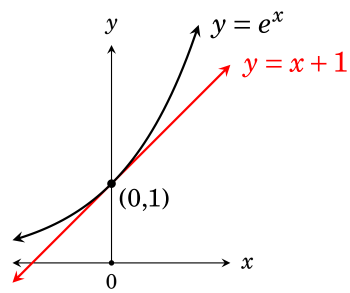


Figure 5.4. The tangent to $y = e^x$ at $(0,1)$ is the line $y = x+1$, with slope 1.

There is a simple reason for this. Pick an integer n and consider the exponential function $f(x) = \left(\left(\frac{1}{n}+1\right)^n\right)^x$, that is, the exponential function whose base is the number $\left(\frac{1}{n}+1\right)^n$. For large n this base is close to e, so we can regard $f(x)$ as being an approximation of e^x. Let $g(x) = x+1$ be the linear function whose graph is the tangent line in Figure 5.4. Since $f(0) = 1 = g(0)$ and $f\left(\frac{1}{n}\right) = \frac{1}{n}+1 = g\left(\frac{1}{n}\right)$ (check this!), the graphs of f and g intersect twice, once at $x = 0$ but also again at $x = \frac{1}{n}$. (See Figure 5.5.) In particular, the line $g(x) = x+1$ is **not** tangent to the graph of the exponential function $y = \left(\left(\frac{1}{n}+1\right)^n\right)^x$. But, as we will develop on the next page, it is **almost** tangent if n is a large number.

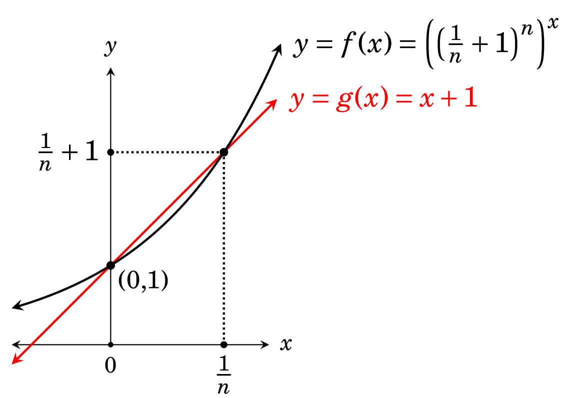
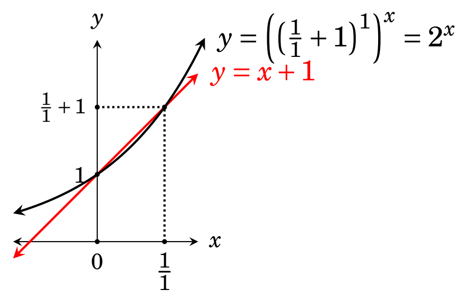
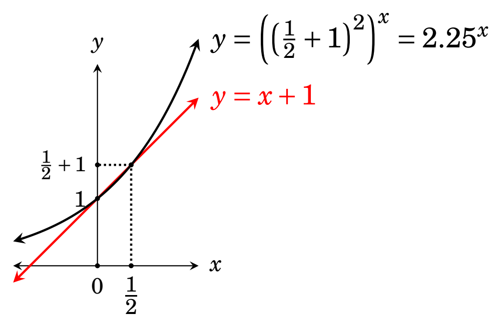


Figure 5.5. The graphs of $y = x+1$ and $y = \left(\left(\frac{1}{n}+1\right)^n\right)^x$ cross at $x=0$ and $x=\frac{1}{n}$.

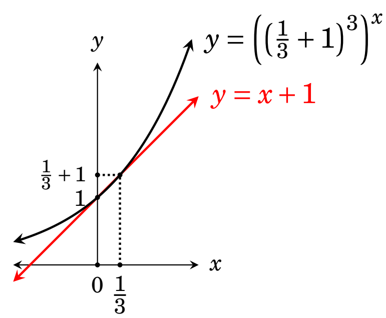
For example, consider the case $n = 1$, for which $\left(\left(\frac{1}{n}+1\right)^n\right)^x = \left(\left(\frac{1}{1}+1\right)^1\right)^x = 2^x$, which is graphed on the right. This exponential function crosses the line $y = x+1$ at $x=0$ and $x = \frac{1}{n} = \frac{1}{1} = 1$. The line $y = x+1$ is definitely not tangent to the exponential function 2^x.



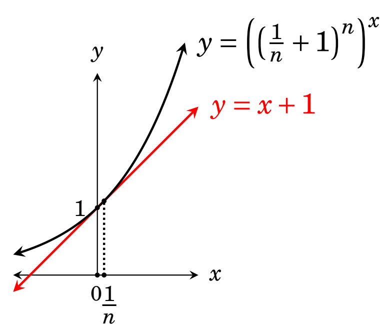
Next consider the case $n = 2$, for which $\left(\left(\frac{1}{n}+1\right)^n\right)^x = \left(\left(\frac{1}{2}+1\right)^2\right)^x = 2.25^x$, which is graphed on the right. This exponential function crosses the line $y = x+1$ at $x=0$ and $x = \frac{1}{n} = \frac{1}{2}$. The intersection points are closer than they were for $n = 1$, but $y = x+1$ is not tangent to the graph of $y = 2.25^x$.



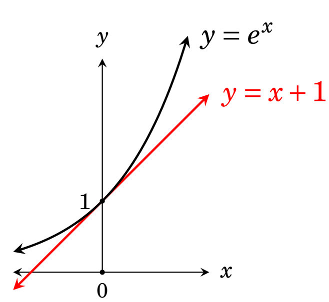
Now consider the case $n = 3$, for which $\left(\left(\frac{1}{n}+1\right)^n\right)^x = \left(\left(\frac{1}{3}+1\right)^3\right)^x$. This exponential function crosses the line $y = x+1$ at $x=0$ and $x = \frac{1}{3}$. The line is still not a tangent, but it is closer to being tangent than is was before.



In general, $y = \left(\left(\frac{1}{n}+1\right)^n\right)^x$ crosses the line $y = x+1$ at $x = 0$ and $x = \frac{1}{n}$. These two intersection points are very close together for large n. Thus when n is large, the line $y = x+1$ is almost—but not quite—tangent to $y = \left(\left(\frac{1}{n}+1\right)^n\right)^x$.

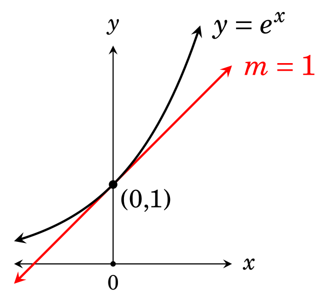


From this line of reasoning, we see that the larger and larger n is, the closer and closer the line $y = x+1$ is to being tangent to $y = \left(\left(\frac{1}{n}+1\right)^n\right)^x$. Because $\left(\frac{1}{n}+1\right)^n$ approaches e as n goes to ∞, we conclude that the line $y = x+1$ is tangent to the graph of $y = e^x$.



In conclusion, the line $y = x + 1$ is the tangent line to the graph of $y = e^x$ at $(0,1)$. This line has slope 1. Let us summarize this.

Fact 5.2 The tangent line to the graph of $y = e^x$ at $(0,1)$ has slope 1.



We will return to this fundamental fact in Chapter 19, where its full significance will emerge. It will be a key ingredient to an important formula.

5.7 Solutions for Chapter 5

Exercises for Section 5.1

1. $25^{1/2} = \sqrt{25} = 5$

3. $\frac{1}{4}^{1/2} = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$

5. $(-27)^{1/3} = \sqrt[3]{-27} = -3$

7. $(-27)^{4/3} = \sqrt[3]{-27}^4 = (-3)^4 = 81$

9. $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

11. $\frac{1}{2}^{-1} = 2$

13. $\frac{1}{2}^{-3} = \frac{1}{(1/2)^3} = 8$

15. $\sqrt{2}^6 = \left(2^{1/2}\right)^6 = 2^3 = 8$

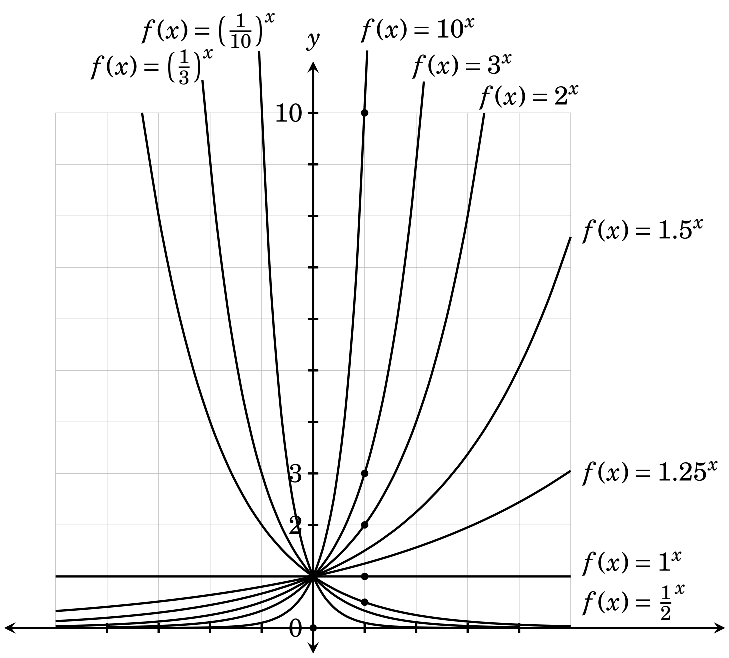
17. $\left(\dfrac{\sqrt{3}}{2}\right)^{-4} = \dfrac{1}{\frac{\sqrt{3}^4}{3^4}} = \dfrac{4}{9}$

19. $\left(\left(\frac{2}{3}\right)^{\frac{3}{2}}\right)^2 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

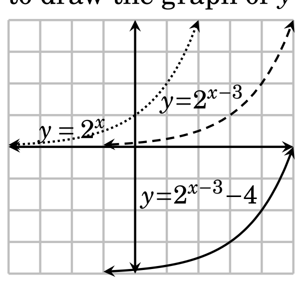
21. $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$

Solutions for Section 5.2

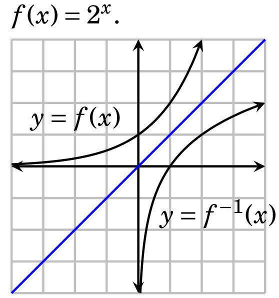
1. Add the graphs of exponential functions $f(x) = \left(\frac{1}{10}\right)^x$ and $f(x) = \left(\frac{1}{3}\right)^x$ to Figure 5.1.



3. Use graph shifting techniques to draw the graph of $y = 2^{x-3} - 4$.



5. Draw the graph of the inverse of $f(x) = 2^x$.



Solutions for Section 5.3

1. $\log_3(81) = 3^{\square}(3^4) = 4$

3. $\log_3(\sqrt{3}) = 3^{\square}(3^{1/2}) = \frac{1}{2}$

5. $\log_3(1) = 3^{\square}(1) = 0$

7. $\log(\sqrt[3]{10}) = 10^{\square}(10^{1/3}) = \frac{1}{3}$

9. $\log(0.01) = 10^{\square}(10^{-2}) = -2$

11. $\log_4(4) = 4^{\square}(4) = 1$

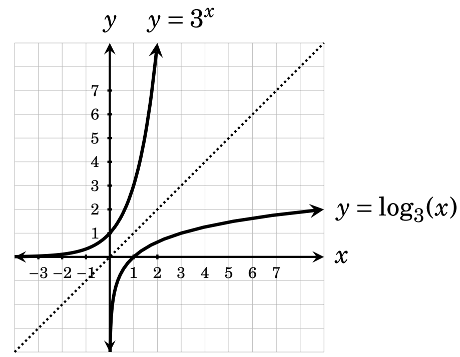
13. $\log_4(\sqrt{2}) = \frac{1}{4}$

15. $\log_4 8 = 4^{\square}(64^{\frac{1}{2}}) = 4^{\square}\left((4^3)^{\frac{1}{2}}\right) = 4^{\square}\left(4^{\frac{3}{2}}\right) = \frac{3}{2}$

17. $10^{\log(5)} = 5.$

19. Draw the graph of $y = 3^x$ and its inverse $y = \log_3(x)$.

See right.



Solutions for Section 5.4

1. $5\log_2(x^3+1) + \log_2(x) - \log_2(3) = \log_2\left((x^3+1)^5\right) + \log_2\left(\frac{x}{3}\right) = \boxed{\log_2\left(\frac{x(x^3+1)^5}{3}\right)}$

3. $2 + \log(5) + 2\log(7) = \log(100) + \log(5) + 2\log(7)$
$= \log(100) + \log(5) + \log(7^2) = \log(500 \cdot 7^2) = \boxed{\log(24500)}$

5. $\log_2\sqrt{\frac{x^3}{x+1}} = \log_2\left(\frac{x^3}{x+1}\right)^{\frac{1}{2}} = \frac{1}{2}\log_2\left(\frac{x^3}{x+1}\right) = \frac{1}{2}\left(\log_2\left(x^3\right) - \log_2(x+1)\right)$
$= \boxed{\frac{3}{2}\log_2(x) - \frac{1}{2}\log_2(x+1)}$

7. $\log\left(\sqrt{x}(x+3)^6\right) = \log\left(\sqrt{x}\right) + \log\left((x+3)^6\right) = \log\left(x^{1/2}\right) + \log\left((x+3)^6\right)$
$= \boxed{\frac{1}{2}\log(x) + 6\log(x+3)}$

9. $\log\left(10x^{10}\right) = \log(10) + \log\left(x^{10}\right) = \boxed{1 + 10\log(x)}$

11. $\log(2) + \log(2x) + \log(25x) = \log(2 \cdot 2x \cdot 25x) = \log(100x^2) = \log(100) + \log(x^2) =$
$\boxed{2 + 2\log(x)}$

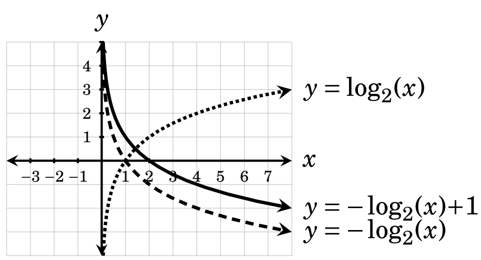
13. Find the inverse of $f(x) = 7 - 2^{x^3+4}$

$$\begin{aligned} y &= 7 - 2^{x^3+4} \\ x &= 7 - 2^{y^3+4} \\ 7 - x &= 2^{y^3+4} \\ \log_2(7-x) &= \log_2\left(2^{y^3+4}\right) \\ \log_2(7-x) &= y^3 + 4 \\ \log_2(7-x) - 4 &= y^3 \\ \sqrt[3]{\log_2(7-x) - 4} &= y \\ f^{-1}(x) &= \sqrt[3]{\log_2(7-x) - 4} \end{aligned}$$

15. Find the inverse of $g(\theta) = \dfrac{2}{5^\theta} + 1$

$$\begin{aligned} y &= \frac{2}{5^\theta} + 1 \\ \theta &= \frac{2}{5^y} + 1 \\ \theta - 1 &= \frac{2}{5^y} \\ \frac{2}{\theta - 1} &= 5^y \\ \log_5\left(\frac{2}{\theta-1}\right) &= \log_5\left(5^y\right) \\ \log_5\left(\frac{2}{\theta-1}\right) &= y \\ g^{-1}(\theta) &= \log_5\left(\frac{2}{\theta-1}\right) \end{aligned}$$

17. Use graph shifting to sketch the graph of $y = -\log_2(x) + 1$.



Exercises for Section 5.5

1. $\ln\left(\dfrac{1}{e}\right) = e^{\square}(e^{-1}) = -1$ **3.** $e^{\sqrt{2}} \cdot e^{-\sqrt{2}} = \dfrac{e^{\sqrt{2}}}{e^{\sqrt{2}}} = 1$ **5.** $e^{\ln(3)-\ln(2)} = \dfrac{e^{\ln(3)}}{e^{\ln(2)}} = \dfrac{3}{2}$

7. Solve $2^{x+3} = 3^x$

$$\begin{aligned} 2^{x+3} &= 3^x \\ \ln\left(2^{x+3}\right) &= \ln\left(3^x\right) \\ (x+3)\ln(2) &= x\ln(3) \\ x\ln(2) + 3\ln(2) &= x\ln(3) \\ x\ln(2) - x\ln(3) &= -3\ln(2) \\ x(\ln(2) - \ln(3)) &= -3\ln(2) \\ x &= \frac{-3\ln(2)}{\ln(2) - \ln(3)} \end{aligned}$$

9. Solve $e^{4x-2} = 3e^2$

$$\begin{aligned} e^{4x-2} &= 3e^2 \\ \ln\left(e^{4x-2}\right) &= \ln\left(3e^2\right) \\ 4x - 2 &= \ln(3) + \ln\left(e^2\right) \\ 4x - 2 &= \ln(3) + 2 \\ 4x &= \ln(3) + 4 \\ x &= \frac{\ln(3) + 4}{4} \end{aligned}$$

11. Solve $e^{\sin(x)} = e$

$$\begin{aligned} e^{\sin(x)} &= e \\ \ln\left(e^{\sin(x)}\right) &= \ln(e) \\ \sin(x) &= 1 \\ x &= \frac{\pi}{2} + k\pi \text{ (k an integer)} \end{aligned}$$

13. Solve $10^x = 17$

$$\begin{aligned} 10^x &= 17 \\ \log(10^x) &= \log(17) \\ x\log(10) &= \log(17) \\ x &= \log(17) \end{aligned}$$

15. $\log_2(5) = \dfrac{\ln(5)}{\ln(2)}$

17. $\log_4(5) = \dfrac{\ln(5)}{\ln(4)}$

19. $\log(8) = \dfrac{\ln(8)}{\ln(10)}$

21. $\log(10) = \dfrac{\ln(10)}{\ln(10)} = 1$

23. $\log_3(8) = \dfrac{\ln(8)}{\ln(3)}$

25. $\log_3(10) = \dfrac{\ln(10)}{\ln(3)}$