CHAPTER 42

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus provides a link between definite integrals and antiderivatives. (That is, between *definite* integrals and *indefinite* integrals.)

Recall that we defined the definite integral of a function f from a to b as

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x$$

(where $\Delta x = (b-a)/n$ and $x_k = a + k\Delta x$). But we mostly avoided the unpleasant chore of working out such limits. As we will see, the fundamental theorem of calculus gives the value of this limit with an almost unbelievably simple expression. It is simply

$$\int_a^b f(x)\,dx \;=\; F(b)-F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

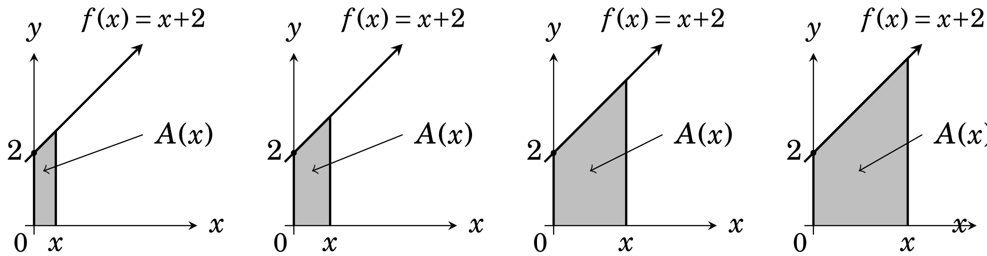
This chapter's purpose is to explain why this is true, and to give examples of computing definite integrals this way.

Our first task in accomplishing this is a minor but necessary detail: Recall that although its notation contains a variable, the definite integral $\int_a^b f(x)dx$ is a *number*. It has the same value no matter what variable we choose. Thus $\int_a^b f(x)dx = \int_a^b f(w)dw = \int_a^b f(t)\,dt$, etc. For this reason, the variable x (or w, or t, etc.) in $\int_a^b f(x)dx$ is called a **dummy variable**.

In our discussions here, we will often want to reserve the variable x for a different purpose. So we will write some of our integrals as $\int_a^b f(t)dt$.

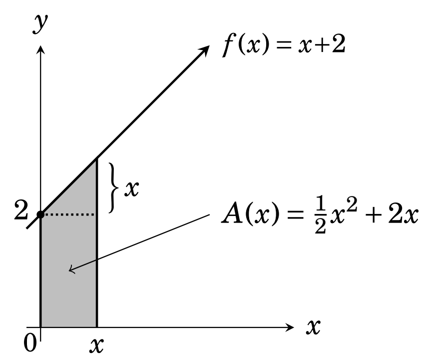
As definite integrals can give area, the expression $\int_a^b f(x)\,dx = F(b)-F(a)$ mentioned above seems to suggest that the area under the graph of $f(x)$ has something to do with *antiderivatives* of f. We will first look at a motivational example that clarifies this. This will then lead to our formulation of the fundamental theorem of calculus. Actually, this theorem has two parts, Part 1 and Part 2. Our example will lead to Part 1, which in turn implies Part 2. (It is Part 2 that says $\int_a^b f(x)\,dx = F(b)-F(a)$.)

Our motivational example is this: Consider the function $f(x) = x+2$, which is a line with slope 1 and y-intercept 2. Take a value of x and consider the region under $y = f(x)$ and between 0 and x. Depending on the value of x, this region has varying widths and thus varying area.



As the area of this region depends on x, the area is a function $A(x)$ of x. For a given value of x, the region can be divided into a triangle with base x and height x, on top of rectangle with base x and height 2. Thus $A(x) =$ (triangle area)+ (rectangle area) = $\frac{1}{2}x\cdot x + 2\cdot x$, or

$$A(x) = \frac{1}{2}x^2 + 2x \text{ square units.}$$



There is an interesting relationship between the function $f(x) = x+2$ and the area $A(x) = \frac{1}{2}x^2 + 2x$ under its graph. The derivative of $A(x)$ is $f(x)$. Thus the area under $f(x)$ is an antiderivative of $f(x)$.

Function	Area under function
$f(x) = x+2$	$A(x) = \frac{1}{2}x^2 + 2x$

D_x

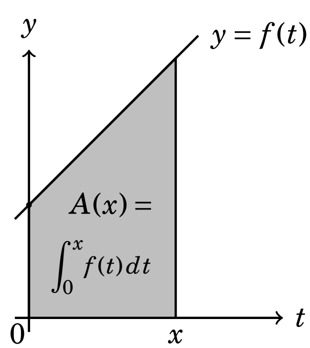
So we have just shown that

$$D_x\Big[A(x)\Big] = f(x).$$

And since $A(x) = \int_0^x f(t)\,dt$, the above can be written as

$$D_x\left[\int_0^x f(t)\,dt\right] = f(x).$$

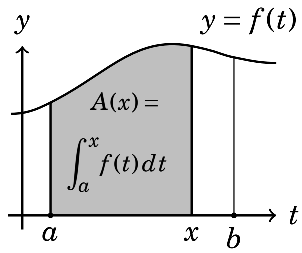
This is not a coincidence. It the fundamental theorem of calculus.



42.1 The Fundamental Theorem of Calculus, Part 1

The example on the previous page is an illustration of Part 1 of the Fundamental Theorem of Calculus.

The general picture is this: Suppose a function f is continuous on an interval $[a,b]$. Then for any x in $[a,b]$ the value of the integral $\int_a^x f(t)\,dt$ is a number that depends on x. Therefore we have a function $A(x) = \int_a^x f(t)\,dt$. (We use t as a dummy variable here because x appears in a different context, as the upper limit of integration.)



For example, $A(a) = \int_a^a f(t)\,dt = 0$. If $f(t)$ happens to be positive on $[a,b]$, then $A(x) = \int_a^x f(t)\,dt$ is the area under $y = f(t)$ and between $t = a$ and $t = x$. However, $f(t)$ need not be positive, and in such a case we can regard $A(x) = \int_a^x f(t)\,dt$ as $A_{\text{up}} - A_{\text{down}}$ between a and x.

Part 1 of the fundamental theorem of calculus simply gives the derivative of this function $A(x)$. It states that $A'(x) = f(x)$.

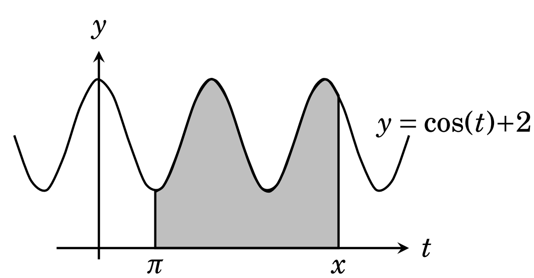
Theorem 42.1 (The Fundamental Theorem of Calculus, Part 1)

Suppose a function $f(x)$ is continuous on the interval $[a,b]$.
Then the function $A(x) = \int_a^x f(t)\,dt$ is differentiable on (a,b).
Its derivative is

$$D_x\left[\int_a^x f(t)\,dt\right] = f(x).$$

Example 42.1 Find the derivative of the function $A(x) = \int_\pi^x \cos(t) + 2\,dt$.

The function $A(x) = \int_\pi^x \cos(t)+2\;dt$ gives the area under the graph of $f(t) = \cos(t) + 2$, between $t = \pi$ and $t = x$. The fundamental theorem of calculus part 1 says is derivative is $D_x\left[\int_\pi^x f(t)\,dt\right] = f(x)$, that is,



$$D_x\left[\int_\pi^x \cos(t) + 2\,dt\right] = \cos(x) + 2.$$

In this example $A(x)$ is area under the graph of $f(x) = \cos(x)+2$, and $A'(x) = f(x) = \cos(x)+2$. This illustrates a general principle. The derivative of area under $f(x)$ is $f(x)$. That is, *area under f is an antiderivative of f* ✍

We will prove part 1 of the fundamental theorem of calculus in Section 42.3. For now we investigate one of its most important consequences – the fundamental theorem of calculus, part 2.

42.2 The Fundamental Theorem of Calculus, Part 2

Part 2 of fundamental theorem of calculus is incredibly useful, as it will give a simple formula for $\int_a^b f(x)\,dx$. It also follows quickly from Part 1, as follows: Suppose a function $f(x)$ is continuous on a closed interval $[a,b]$. Part 1 of the fundamental theorem says

$$D_x\left[\int_a^x f(t)\,dt\right] = f(x). \qquad (*)$$

This means that the function $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. Let $F(x)$ be *any* antiderivative of $f(x)$. Then $F(x)$ and $\int_a^x f(t)\,dt$ are both antiderivatives of $f(x)$, so they differ by a constant C:

$$\int_a^x f(t)\,dt = F(x)+C.$$

We can actually find C by inserting $x = a$:

$$0 = \int_a^a f(t)\,dt = F(a)+C.$$

Therefore $C = -F(a)$, and Equation $(*)$ becomes

$$\int_a^x f(t)\,dt = F(x)-F(a).$$

Inserting $x = b$ into this, we get

$$\int_a^b f(t)\,dt = F(b)-F(a).$$

Now the x is gone, so the dummy variable t can be replaced with x:

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

This is our formula for $\int_a^b f(x)\,dx$, and in fact it is part 2 of the fundamental theorem of calculus.

> **Theorem 42.2** (Fundamental Theorem of Calculus, Part 2)
> If f is continuous on $[a,b]$, and F is any antiderivative of f, then
> $$\int_a^b f(x)\,dx \;=\; F(b)-F(a)$$

Example 42.2 Find $\displaystyle\int_0^2 x^2\,dx$.

Solution Part 2 of the fundamental theorem of calculus says this equals $F(2)-F(0)$, where F is any antiderivative of $f(x)=x^2$. The antiderivatives of x^2 are $F(x)=\int x^2\,dx=\frac{1}{3}x^3+C$. We are allowed to use *any* antiderivative of x^2, so we may as well put $C=0$, so $F(x)=\frac{1}{3}x^3$. Then

$$\int_0^2 x^2\,dx \;=\; F(2)-F(0) \;=\; \frac{1}{3}2^3-\frac{1}{3}0^3 \;=\; \frac{8}{3}.$$ ✍

Comment: If we had used $F(x)=\frac{1}{3}x^3+C$ in this computation, then the C's would cancel out in $F(2)-F(0)$. So in using the fundamental theorem of calculus, we can simplify our work by always choosing $C=0$.

Compare Example 42.2 with Example 2.2, where we used a limit of Riemann sums to find the area under $f(x)=x^2$ between $x=0$ and $x=2$. The answer came after two pages of work: 8/3 square units. The fundamental theorem of calculus gave this answer instantly in Example 42.2.

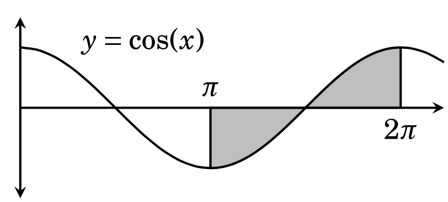
Example 42.3 Find $\displaystyle\int_1^4 \sqrt{x}+2\,dx$.

Solution Part 2 of the fundamental theorem of calculus says this equals $F(2)-F(0)$, where $F(x)=\int\sqrt{x}+2\,dx=\frac{2}{3}\sqrt{x}^3+2x+C$. Using $C=0$,

$$\int_1^4 \sqrt{x}+2\,dx \;=\; F(4)-F(1) \;=\; \left(\frac{2}{3}\sqrt{4}^3+2\cdot 4\right)-\left(\frac{2}{3}\sqrt{1}^3+2\cdot 1\right) \;=\; \frac{32}{3}.$$ ✍

Example 42.4 Find $\displaystyle\int_\pi^{2\pi} \cos(x)\,dx$.

Solution As $\int\cos(x)\,dx=\sin(x)+C$, the fundamental theorem gives $\int_\pi^{2\pi}\cos(x)dx=\sin(\pi)-\sin(2\pi)=0-0=0$. This makes sense because the areas above and below the x-axis cancel. ✍



Part 2 of the fundamental of calculus is used so frequently that we usually just call it the fundamental theorem of calculus, or FTC.

The difference $F(b)-F(a)$ that appears in the FTC is so prevalent that we have a special abbreviation for it:

$$F(b)-F(a)=\big[F(x)\big]_a^b.$$

For example, $\big[\cos(x)\big]_0^\pi=\cos(\pi)-\cos(0)=-1-1=-2$. In other texts you may see $\big[F(x)\big]_a^b$ written as $F(x)\Big|_a^b$ or $F(b)-F(a)=F(x)\Big]_a^b$. With this new notation we can state the FTC as follows:

Fundamental Theorem of Calculus, Part 2 (FTC):
$\displaystyle\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b$ (where F is an antiderivative of f).

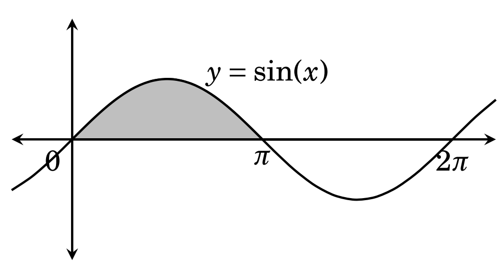
Example 42.5 $$\int_{-1}^{1}\left(x^5-x+1\right)\,dx=\left[\frac{x^6}{6}-\frac{x^2}{2}+x\right]_{-1}^{1}=\left(\frac{1^6}{6}-\frac{1^2}{2}+1\right)-\left(\frac{(-1)^6}{6}-\frac{(-1)^2}{2}-1\right)=2.$$ ✍

Example 42.6 $$\int_0^1\frac{1}{1+x^2}dx=\Big[\tan^{-1}(x)\Big]_0^1=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}-0=\frac{\pi}{4}.$$ ✍

Example 42.7 $$\int_1^5\frac{1}{x}\,dx=\Big[\ln(x)\Big]_1^5=\ln(5)-\ln(1)=\ln(5)-0=\ln(5).$$ ✍

Example 42.8 Find the area of the region under the graph of $\sin(x)$, above the x-axis and between $x=0$ and $x=\pi$.

Solution Because $\sin(x)\geq 0$ on $[0,\pi]$, the area in question is

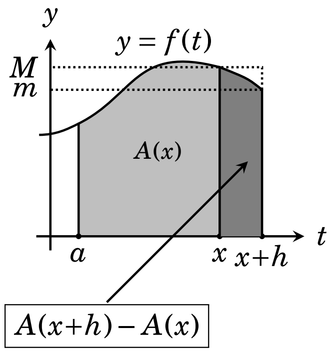


$$\begin{aligned}\int_0^\pi \sin(x)\,dx &= \Big[-\cos(x)\Big]_0^\pi\\ &= -\cos(\pi)-(-\cos(0))\\ &= -(-1)-(-1)\\ &= 2\text{ square units.}\end{aligned}$$ ✍

Now test your understanding by working some exercises.

42.3 Proof of the Fundamental Theorem

[I'll add a proof here soon. For now, see the explanation in Lecture 43.]



Exercises for Chapter 42

1. $\displaystyle\int_0^2 \left(3x-4x^3\right)\,dx =$

2. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos(x)\,dx =$

3. $\displaystyle\int_0^9 \sqrt{x}\,dx =$

4. $\displaystyle\int_{-1}^{1} (x^2+x+1)\,dx =$

5. $\displaystyle\int_1^{e^2} \frac{1}{x}\,dx =$

6. $\displaystyle\int_1^2 \left(1+\frac{1}{x^2}\right)\,dx =$

7. $\displaystyle\int_{-8}^{8} 5\sqrt[3]{x}\,dx =$

8. $\displaystyle\int_0^8 5\sqrt[3]{x}\,dx =$

9. $\displaystyle\int_0^{\pi/3} \sec^2(x)\,dx =$

10. $\displaystyle\int_1^2 x\left(x+\frac{1}{x}\right)\,dx =$

11. $\displaystyle\int_0^{\pi/3} \sec(x)\tan(x)\,dx =$

12. $\displaystyle\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx =$

13. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx =$

14. $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx =$

15. $\displaystyle\int_1^4 \frac{5x^2+1}{x^2}\,dx =$

16. $\displaystyle\int_1^3 \frac{x^5-8x}{x^3}\,dx =$

17. $\displaystyle\int_1^4 x^{-\frac{1}{2}}\,dx =$

18. $\displaystyle\int_{\pi/4}^{3\pi/4} \cot x \csc x\,dx =$

19. Find the area under the graph of $y=\sqrt{x}$ between $x=4$ and $x=9$.

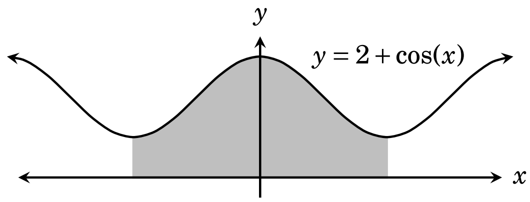
20. Find the area under the graph of $y=\sqrt[3]{x^2}$ between $x=-1$ and $x=8$.

21. Find the area under the graph of $y=\cos(x)$ between $x=0$ and $x=\pi/3$.

22. Find the area under the graph of $y=1/x$ between $x=1/e$ and $x=e$.

23. Find the area under the graph of $y=\sec^2(x)$ between $x=0$ and $x=\pi/4$.

24. Find the area of the shaded region indicated below.



Exercise Solutions for Chapter 42

1. $\int_0^2 \left(3x-4x^3\right)\,dx = \left[\frac{3}{2}x^2 - x^4\right]_0^2 = \left(\frac{3}{2}2^2-2^4\right)-\left(\frac{3}{2}0^2-0^4\right) = -10$

3. $\int_0^9 \sqrt{x}\,dx = \int_0^9 x^{1/2}\,dx = \left[\frac{1}{1/2+1}x^{1/2+1}\right]_0^9 = \left[\frac{2}{3}x^{3/2}\right]_0^9 = \left[\frac{2}{3}\sqrt{x}^3\right]_0^9 = \frac{2}{3}\sqrt{9}^3 = \frac{54}{3}$

5. $\int_1^{e^2} \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^{e^2} = \ln|e^2| - \ln|1| = 2-0 = 2$

7. $\int_{-8}^8 5\sqrt[3]{x}\,dx = \int_{-8}^8 5x^{1/3}\,dx = \left[5\frac{1}{1/3+1}x^{1/3+1}\right]_{-8}^8 = \left[\frac{15}{4}x^{4/3}\right]_{-8}^8 = \left[\frac{15}{4}\sqrt[3]{x}^4\right]_{-8}^8$

$$= \frac{15}{4}\sqrt[3]{8}^4 - \frac{15}{4}\sqrt[3]{-8}^4 = \frac{15}{4}(2)^4 - \frac{15}{4}(-2)^4 = 0$$

9. $\int_0^{\pi/3} \sec^2(x)\,dx = \Big[\tan(x)\Big]_0^{\pi/3} = \tan(\pi/3)-\tan(0) = \sqrt{3}-0 = \sqrt{3}$

11. $\int_0^{\pi/3} \sec(x)\tan(x)\,dx = \Big[\sec(x)\Big]_0^{\pi/3} = \sec(\pi/3)-\sec(0) = 2-1 = 1$

13. $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \Big[\sin^{-1}(x)\Big]_{-1}^1 = \sin^{-1}(1)-\sin^{-1}(-1) = \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \pi$

15. $\int_1^4 \frac{5x^2+1}{x^2}\,dx = \int_1^4 \left(\frac{5x^2}{x^2}+\frac{1}{x^2}\right)\,dx = \int_1^4 \left(5+\frac{1}{x^2}\right)\,dx = \left[5x-\frac{1}{x}\right]_1^4 = \left(20-\frac{1}{4}\right)-(5-1) = \frac{63}{4}$

17. $\int_1^4 x^{-\frac{1}{2}}\,dx = \left[\frac{1}{-1/2+1}x^{\frac{1}{2}}\right]_1^4 = \Big[2\sqrt{x}\Big]_1^4 = 2\sqrt{4}-2\sqrt{1} = 2$

19. Find the area under the graph of $y=\sqrt{x}$ between $x=4$ and $x=9$.

$\int_4^9 \sqrt{x}\,dx = \int_0^9 x^{1/2}\,dx = \left[\frac{1}{1/2+1}x^{1/2+1}\right]_4^9 = \left[\frac{2}{3}x^{3/2}\right]_4^9 = \left[\frac{2}{3}\sqrt{x}^3\right]_4^9 = \frac{2}{3}\sqrt{9}^3 - \frac{2}{3}\sqrt{4}^3 = \frac{38}{3}$ square units

21. Find the area under the graph of $y=\cos(x)$ between $x=0$ and $x=\frac{\pi}{3}$.

$\int_0^{\pi/3} \cos(x)\,dx = \Big[\sin(x)\Big]_0^{\pi/3} = \sin(\pi/3)-\sin(0) = \frac{\sqrt{3}}{2}-0 = \frac{\sqrt{3}}{2}$ square units

23. Find the area under the graph of $y=\sec^2(x)$ between $x=0$ and $x=\pi/4$.

$\int_0^{\pi/4} \sec^2(x)\,dx = \Big[\tan(x)\Big]_0^{\pi/4} = \tan(\pi/4)-\tan(0) = 1-0 = 1$ square unit.