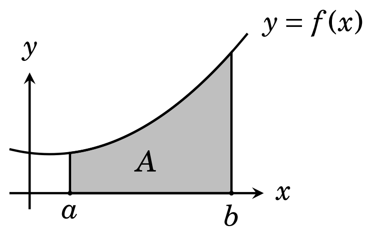
CHAPTER 41

Definite Integrals

This chapter introduces a major concept in calculus, the *definite integral*. In terms of significance, it is as important a concept as the derivative. The previous chapter's area formula is a gateway to this major concept.



Recall that if $f(x) > 0$ on the interval $[a,b]$, then the area of the region (shown above) over $[a,b]$ but below the graph of $y = f(x)$ is given by the limit

$$\text{Area} \;=\; A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x,$$

where for any n, $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$.

This area formula is a limit of the special sum of form

$$\sum_{k=1}^{n} f(x_k)\,\Delta x,$$

which we interpreted as a sum of areas of rectangles. In general, a sum having this form is called a **Riemann sum**.

We often think of a Riemann sum as being the sum of the areas of n rectangles, which (for large n) approximates the area A under $f(x)$. However, Riemann sums do not necessarily approximate just area. In fact they can be negative. If $f(x)$ is ever negative on $[a,b]$, then some (or all) of the terms $f(x_k)\Delta x$ in the Riemann sum can be negative. (Think of them as "*rectangles with negative height*.") See Figure 41.1. Note that any rectangles under the x-axis give a negative contribution to the sum $\sum_{k=1}^{n} f(x_k)\,\Delta x$.

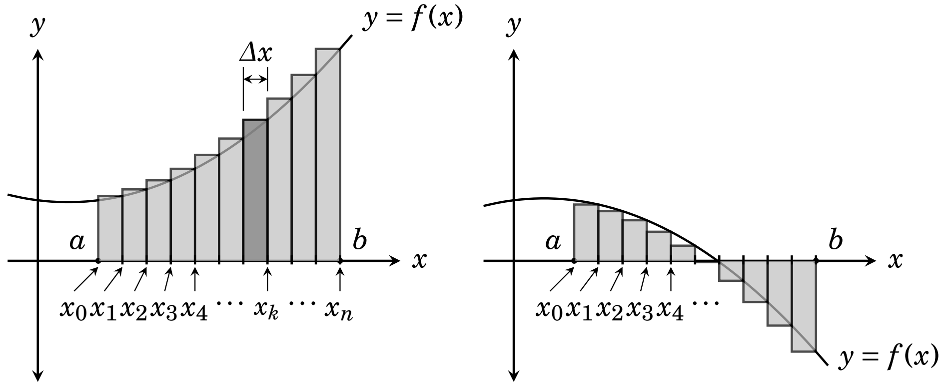


Figure 41.1. In a Riemann sum $\sum_{k=1}^{n} f(x_k)\Delta x$, any rectangles below the x-axis give negative contributions to the sum. So it is possible for a Riemann sum to be negative, or zero.

Now we are ready for our big definition. Given a function $f(x)$ defined on an interval $[a,b]$, its *definite integral* is a certain *number*, which we will denote as $\int_a^b f(x)\,dx$. This number is given by our area formula.

> **Definition 41.1** Given a function $f(x)$ defined on a closed interval $[a,b]$, the **definite integral** of $f(x)$ from a to b is the number, denoted as $\int_a^b f(x)\,dx$ (read "*the integral from a to b* $f(x)$ dx"), and defined as
>
> $$\int_a^b f(x)\,dx \quad = \quad \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$
>
> (provided the limits exists), where for any n, $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta a$. If the limit does not exist, then we say the definite integral does not exist.

Since we have used our area formula in the definition of the definite integral, it follows that if $f(x) \geq 0$ on $[a,b]$, the definite integral $\int_a^b f(x)\,dx$ equals the area over $[a,b]$ and under the graph of $y = f(x)$.

In Example 40.2 we computed the area over $[0,2]$ and under $y = f(x) = x^2$, and found that this region contains 8/3 square units of area. Therefore

$$\int_0^2 x^2\,dx = \frac{8}{3}.$$

Take note: A **definite integral** is a **number**, for example, $\int_0^2 x^2\,dx = \frac{8}{3}$. An **indefinite integral** is a set of **functions**, for example, $\int x^2\,dx = \frac{x^3}{3} + C$.

There is a good reason why the notations these two things are so similar. The Fundamental Theorem of Calculus (next chapter) will give a formula for the definite integral $\int_a^b f(x)dx$ in terms of the indefinite integral $\int f(x)dx$.

For now, in the definition

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x,$$

regard the $\int_a^b$ on the left as a a streamlined version of $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}$ on the right, whereas the $f(x)$ corresponds to $f(x_k)$, and the dx to Δx.

We will accept on faith the following theorem from advanced calculus, which assures us that the definite integral exists provided f is continuous.

Theorem 41.1 If $f(x)$ is continuous on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx$ exists.

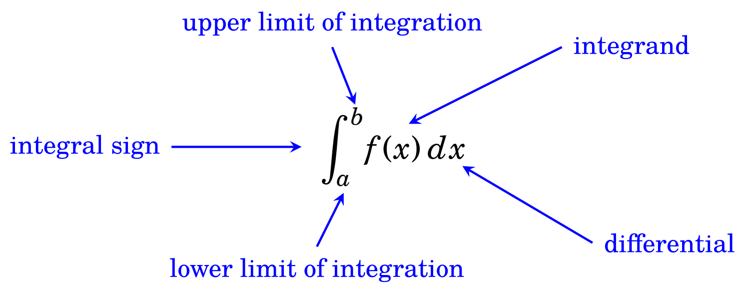
This means that the limit $\lim\limits_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$ in the definition of $\int_a^b f(x)\,dx$ is guaranteed to exist as long as f is continuous on $[a,b]$. However, even though the limit exists, it can be quite difficult to evaluate it directly.

For instance, consider the definite integral of $f(x)=x^2+\cos(x)$ over $[a,b]=[-\pi,\pi]$. Following Definition 41.1, $\Delta x=\frac{\pi-(-\pi)}{n}=\frac{2\pi}{n}$ and $x_k=-\pi+k\Delta x=-\pi+\frac{2\pi k}{n}$. So

$$\begin{aligned}\int_{-\pi}^{\pi} x^2+\cos(x)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(x_k^2+\cos(x_k)\right)\Delta x\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(-\pi+\frac{2\pi k}{n}\right)^2+\cos\left(-\pi+\frac{2\pi k}{n}\right)\right)\frac{2\pi}{n}.\end{aligned}$$

This is not an easy limit! But by Theorem 41.1, it *does* exist. (And the fundamental theorem of calculus, in Chapter 42, will give a quick answer.)

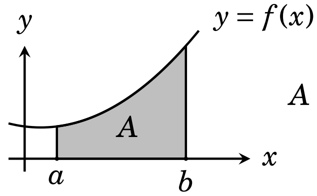
Some vocabulary: In the expression $\int_a^b f(x)\,dx$, the symbol $\int$ is called the **integral sign**, and the function $f(x)$ that is being integrated is called the **integrand**. The numbers a and b are called the **limits of integration**.



41.1 Properties of Definite Integrals

Definite integrals have many properties, and we list eight of them here. The first property involves area.

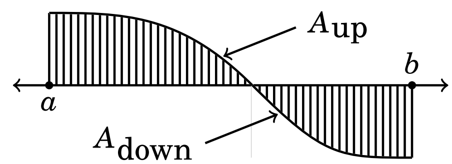
Property 1 (The Area Property) Because we defined definite integrals with our area formula, it follows that if $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ equals the area of the region over $[a,b]$ and below the graph of $y = f(x)$.



$$A = \int_a^b f(x)\,dx$$

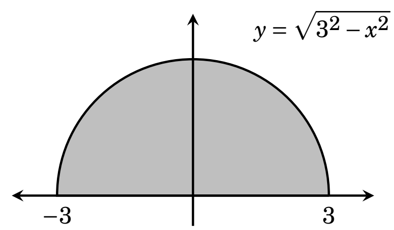
More generally, if $f(x)$ is not positive on all of $[a,b]$ then any rectangles in the Riemann sum $\lim\limits_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$ that are below the x-axis give a negative contribution to area. Thus $\int_a^b f(x)\,dx$ equals the area contained above the x-axis, minus any area below the x-axis. We express this as

$$\int_a^b f(x)\,dx = A_{\text{up}} - A_{\text{down}},$$



where A_{up} is the area above the x-axis and A_{down} is the area below.

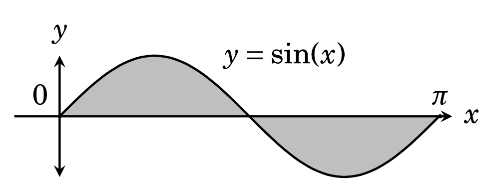
Example 41.1 Find $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$.



The graph of $y = \sqrt{9-x^2} = \sqrt{3^2-x^2}$ is the upper half of the circle of radius 3 centered at the origin. The area under the curve is thus half the area of a circle of radius 3, or $\frac{1}{2}\pi 3^2 = \frac{9\pi}{2}$.

Thus $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx = \boxed{\frac{9\pi}{2}.}$ ✍

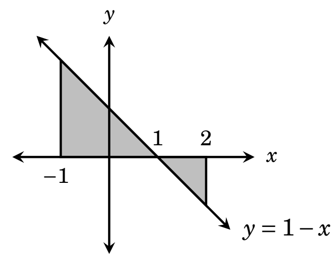
Example 41.2 Find $\int_0^{2\pi} \sin(x)\,dx$.



Here A_{up} and A_{down} are equal, so $\displaystyle\int_0^{2\pi} \sin(x)\,dx = A_{\text{up}} - A_{\text{down}} = \boxed{0.}$ ✍

Example 41.3 Find $\int_{-1}^{2}(1-x)\,dx$.

The graph of $y = 1-x$ is a straight line with slope -1 and y-intercept 1. The region contained between -1 and 2 consists of two triangles, shown below. The area of a triangle is is $\frac{1}{2}bh$, so



$$\begin{aligned}\int_{-1}^{2}(1-x)\,dx &= A_{\text{up}} - A_{\text{down}}\\ &= \frac{1}{2}2\cdot 2 - \frac{1}{2}1\cdot 1\\ &= \boxed{3/2.}\end{aligned}$$

✍

The above examples were easy because the regions were recognizable shapes, with familiar area formulas. This is of course not aways the case.

Let's continue with our list of definite integral properties. The next property is obvious because the area between $x = a$ and $x = a$ has to be zero.

Property 2	$\int_{a}^{a} f(x)\,dx = 0$

Next, a property concerning interchanging the limits of integration.

Property 3	$\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$

To see why this is true, set up the limit for each integral. For the integral on the left, $\Delta x = \frac{b-a}{n}$. But for the integral on the right, $\Delta x = \frac{a-b}{n}$, which is the negative of the Δx on the left. Working the details, we get Property 3.

There is also a constant multiple rule for integrals.

Property 4	$\int_{a}^{b} c\,f(x)dx = c\int_{a}^{b} f(x)dx$	(for any constant c)

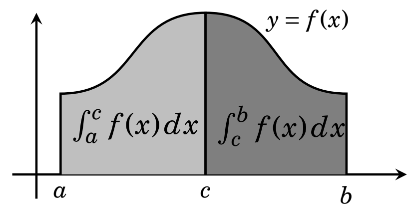
To verify this, set up the limit for $\int_a^b c\,f(x)dx$ and factor the c through the sum and the limit:

$$\int_{a}^{b} c\,f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} c f(x_k)\Delta x = c\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = c\int_{a}^{b} f(x)dx.$$

In a like manner we get a sum-difference rule for definite integrals.

Property 5 $\displaystyle\int_a^b \Big(f(x) \pm g(x)\Big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Property 6 If $a \le c \le b$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



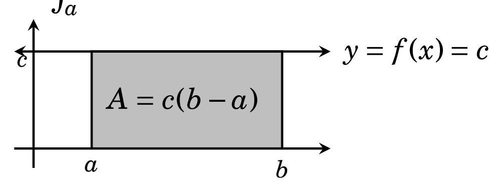
Property 6 is best visualized when $f(x) \ge 0$ on $[a,b]$, so that the integrals give area. In this case $\int_a^c f(x)\,dx$ is the area under $f(x)$ between a and c, and $\int_c^b f(x)\,dx$ is the area under $f(x)$ between c and b. (See the drawing in the box, above.) Add these two areas together, and we get the area under $f(x)$ between a and b, which is $\int_a^b f(x)\,dx$. Thus $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example 41.4 Suppose $\displaystyle\int_2^4 f(x)\,dx = \frac{3}{2}$ and $\displaystyle\int_3^4 f(x)\,dx = \frac{5}{2}$. Find $\displaystyle\int_2^3 f(x)\,dx$.

Solution By Property 6, $\displaystyle\int_2^4 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx$. Therefore $\displaystyle\frac{3}{2} = \int_2^3 f(x)\,dx + \frac{5}{2}$. Then $\displaystyle\int_2^3 f(x)\,dx = \frac{3}{2} - \frac{5}{2} = \boxed{-1}$. ✍

Next a property of the definite integral of a constant function $f(x) = c$.

Property 7 If c is a constant, then $\displaystyle\int_a^b c\,dx = c(b-a)$.



This is clearly true for positive c because the region enclosed between $[a,b]$ and the line $y = c$ is a rectangle of height c and base $b-a$, so its area is $c(b-a)$. (If c is not positive, make the same argument, using A_{down}.)

Example 41.5 Find $\int_{-3}^{3} 2+4\sqrt{9-x^2}\,dx$

$$\begin{aligned}
\int_{-3}^{3} 2+4\sqrt{9-x^2}\,dx &= \int_{-3}^{3} 2\,dx + \int_{-3}^{3} 4\sqrt{9-x^2}\,dx && \text{(Property 5)}\\
&= 2\big(3-(-3)\big) + 4\int_{-3}^{3} \sqrt{9-x^2}\,dx && \text{(Properties 7 \& 4)}\\
&= 12+4\frac{\pi 3^2}{2} = \boxed{12+18\pi} && \text{(by Example 41.1)}
\end{aligned}$$

Our final two properties are less useful for daily computation, but are occasionally needed in proofs, particularly in the next chapter.

Property 8 If $f(x) \le g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

To verify this, suppose $f(x) \le g(x)$ and simply note that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x \;\le\; \lim_{n\to\infty}\sum_{k=1}^{n} g(x_k)\Delta x \;=\; \int_a^b g(x)\,dx,$$

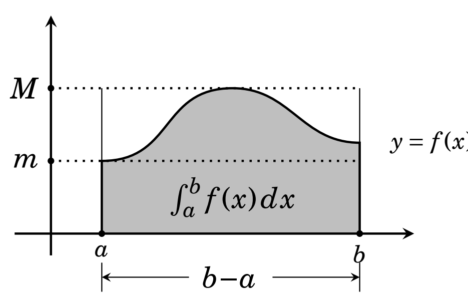
because each $f(x_k)\Delta x$ in the sum on the left is no more than the corresponding $g(x_k)\Delta x$ on the right.

Property 9 If there are two numbers m and M for which $m \le f(x) \le M$ for all x in $[a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

We can verify this using properties 7 and 8, for these properties imply that

$$m(b-a) = \int_a^b m\,dx \;\le\; \int_a^b f(x)\,dx \;\le\; \int_a^b M\,dx \;=\; M(b-a).$$

Property 9 is especially easy to see when $f(x) > 0$ on $[a,b]$. For then the region above $[a,b]$ and below $y = f(x)$ (shaded on the right) contains an $m \times (b-a)$ rectangle. But it is also *contained in* an $M \times (b-a)$ rectangle. As the area of the region is $\int_a^b f(x)\,dx$, Property 9 follows.



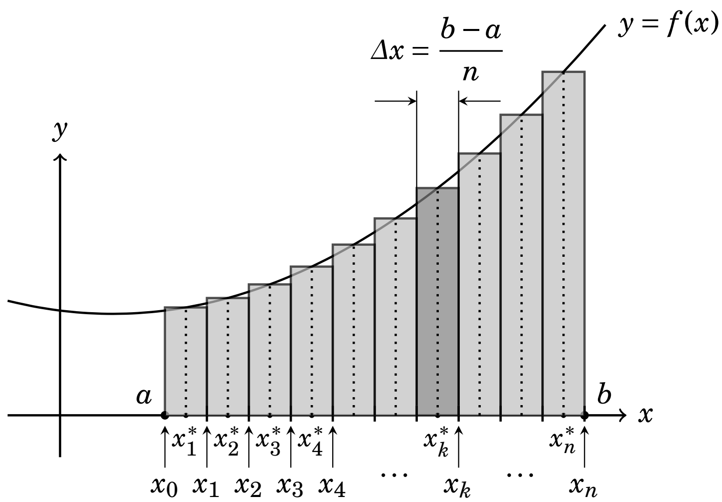
41.2 Other Formulations of the Definite Integral

Our Definition 41.1 is among the simplest and most pared-down of all definitions of the definite integral. This section presents two slightly different definitions. Although they will not used in this text, knowing them can be useful in matching what you've learned from this text with what you may read elsewhere.

The first alternative to Definition 41.1 is really not all that different. To define $\int_a^b f(x)\,dx$, we put $\Delta x = (b-a)/n$, and $x_k = a + k\Delta x$ (for $0 \le k \le n$) just as called for in Definition 41.1. As before this divides the interval $[a,b]$ into subintervals

$$[x_0,x_1],\ \ [x_1,x_2],\ \ [x_2,x_3],\ \ \ldots,\ \ [x_{n-1},x_n],$$

each of length Δx. Next (and this where we diverge from Definition 41.1), in each subinterval $[x_{k-1},x_k]$, select a **sample point** x_k^*. This sample point x_k^* could be anywhere in $[x_{k-1},x_k]$, including all the way to the right (so $x_k^* = x_k$), or all the way to the left ($x_k^* = x_{k-1}$). (See the drawing below.)



On each subinterval $[x_{k-1},x_k]$, establish a rectangle whose base is $[x_{k-1},x_k]$ and whose height is $f(x_k^*)$. The area of this rectangle is then $f(x_k^*)\,\Delta x$ (with the understanding that this could be negative if $f(x_k^*)$ is negative).

Under this set-up (and as illustrated in the drawing above), we define

$$\boxed{\int_a^b f(x)\,dx \ = \ \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x.}$$

Notice that this coincides with Definition 41.1 if in each subinterval the sample point is chosen as far right as possible, so $x_k^* = x_k$.

The second alternate formulation of the definite integral allows for subintervals of varying lengths (so not all subintervals must have the same length Δx). It works as follows.

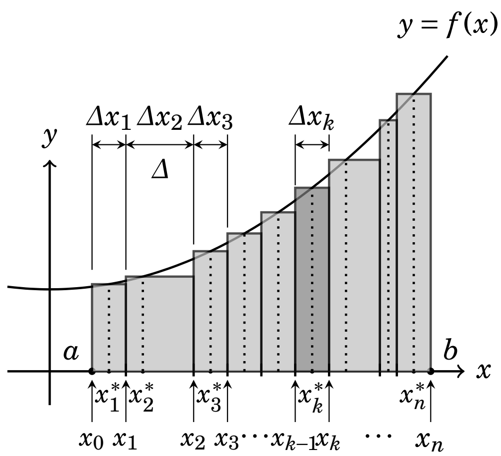
For each positive integer n, divide the interval $[a, b]$ into n subintervals by selecting numbers $x_0 < x_1 < x_2 < \cdots < x_n$ (not necessarily equally spaced) with $a = x_0$ and $b = x_n$. We call this the **partition** for n. This partition divides $[a, b]$ into n subintervals

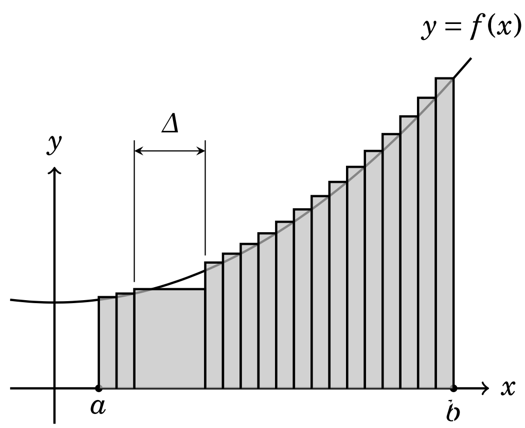
$$[x_0, x_1], \ [x_1, x_2], \ [x_2, x_3], \ \ldots, \ [x_{n-1}, x_n],$$

which may have different lengths. The kth subinterval is $[x_{k-1}, x_k]$, and we denote its length as Δx_k, so

$$\Delta x_k = x_{k-1} - x_k.$$

In this way $[a, b]$ is divided into subintervals of widths $\Delta x_1, \Delta x_2, \ldots, \Delta_n$. In each subinterval $[x_{k-1}, x_k]$, choose a **sample point** x_k^*, as shown below, left.





Now on each $[x_{k-1}, x_k]$ establish a rectangle of height $f(x_k^*)$. This scheme results in n rectangular strips, such that the kth rectangle has height $f(x_k^*)$ and base Δx_k. The Riemann sum is $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$.

At this point it is tempting assert $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$, but there is a problem with this. The number n of rectangles could go to infinity with some of the rectangles remaining thick, while more and more skinny rectangles are squeezes around them. (See drawing, above right.) If this happens, the rectangles don't fit the curve, even if the number of them goes to infinity.

This is overcome as follows. For each partition with n rectangles, let Δ be the largest of the numbers $\Delta x_1, \Delta x_2, \Delta x_3, \dots \Delta x_n$. That is Δ is the width of the thickest rectangle in the Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$.

We call Δ the **norm** of the partition. If we insist that $\Delta \to 0$ as $n \to \infty$, then *all* the rectangles in the Riemann sum get skinnier and skinnier, and therefore fit the curve. This observation leads to our second alternative definition of the definite integral. Given the above setup, we define

$$\boxed{\int_a^b f(x)\,dx \;=\; \lim_{\Delta\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k.}$$

This definition offers the flexibility required for certain practical or theoretical situations, and you may see this formulation in other texts. But—for the purposes of this text—we will use Definition 41.1 exclusively.

Exercises for Chapter 41

1. Using Definition 41.1, write out the integral and limit for the area under the curve $y = \ln(x^3)$ between $x = 1$ and $x = e$.

2. Using Definition 41.1, write out the integral and limit for the area under the curve $y = e^{2x}$ between $x = 0$ and $x = \ln(2)$.

3. Using Definition 41.1, write out the integral and limit for the area under the curve $y = \sin(x)$ between $x = 0$ and $x = \pi$.

4. Using Definition 41.1, write out the integral and limit for the area under the curve $y = x^2 + x + 10$ from $x = 1$ to $x = 5$.

5. Find $\int_0^5 \sqrt{25 - x^2}\,dx$ by considering area.

6. Find $\int_3^5 2x + 10\,dx$ by considering area.

7. Find $\int_{-2}^2 2x + 2\,dx$ by considering area.

8. Find $\int_{-\pi/4}^{\pi/4} \tan(x)\,dx$ by considering area.

9. Find $\int_0^3 |x-2|\,dx$ by considering area.

10. Find $\int_0^3 |2y-6|\,dy$ by considering area.

11. Find $\int_{-2}^4 (|w|-2)\,dw$ by considering area.

12. Find $\int_{-2}^3 f(x)\,dx$, where $f(x) = \begin{cases} x+2 & \text{if} \quad -2 \le x \le 0 \\ \sqrt{4-x^2} & \text{if} \quad 0 \le x \le 2 \\ 2-x & \text{if} \quad 2 \le x \le 3 \end{cases}$

13. Suppose f is a function for which $\int_1^5 f(x)dx = 3$ and $\int_1^7 f(x)dx = -6$. Find $\int_5^7 f(x)dx$.

14. Suppose f is a function for which $\int_2^5 f(x)dx = 7$ and $\int_2^8 f(x)dx = 8$. Find $\int_5^8 f(x)dx$.

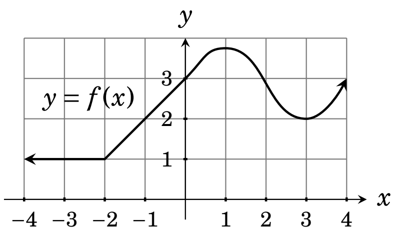
15. Suppose f is a function for which $\int_2^5 f(x)dx = 4$ and $\int_2^8 f(x)dx = 9$. Find $\int_8^5 7f(x)dx$.

16. Suppose f and g are functions for which $\int_0^5 f(x)\,dx = 3$, $\int_0^2 3g(x)\,dx = 12$, and $\int_2^5 g(x)\,dx = -1$. Find $\int_0^5 3f(x) - g(x)\,dx$.

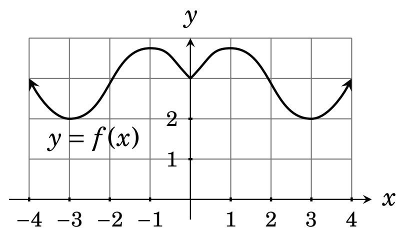
17. Write $\int_0^2 f(x)\,dx - \int_{-2}^2 f(x)\,dx - \int_{-4}^{-2} f(x)\,dx$ as a **single integral** of the form $\int_a^b f(x)\,dx$.

18. Write $\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^1 f(x)\,dx$ as a **single integral** of the form $\int_a^b f(x)\,dx$.

19. A function $f(x)$ is graphed below. If $\int_{-4}^4 f(x)dx = 17.8$, what is $\int_0^4 f(x)dx$?



20. A function $f(x)$ is graphed below. If $\int_{-4}^4 f(x)dx = 22.6$, what is $\int_0^4 f(x)dx$?



21. For a positive integer n, let $\Delta x = 1/n$, and for each integer k (where $1 \le k \le n$), let $x_k = k\Delta x$. Consider the limit $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} x_k \sqrt{x_k^2+1}\,\Delta x$. Write a definite integral in the form $\int_a^b f(x)\,dx$ that equals this limit.

22. Write the limit $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{2+7k/n}{1+(2+7k/n)^2}\,\dfrac{7}{n}$ as a definite integral.

23. Write the limit $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(-1+\dfrac{2k}{n}\right)\cdot\cos\left(\left(-1+\dfrac{2k}{n}\right)^2\right)\dfrac{2}{n}$ as a definite integral.

24. Write the limit $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \sin\left(\sqrt{\dfrac{\pi k}{n}}\right)\dfrac{\pi}{n}$ as a definite integral.

Exercise Solutions Chapter 41

1. Using Definition 41.1, write out the integral and limit for the area under the curve $y = \ln(x^3)$ between $x = 1$ and $x = e$.

The integral is $\displaystyle\int_1^e \ln\left(x^3\right)\,dx$. Here $\Delta x = (e-1)/n$ and $x_k = 1 + k\Delta x = 1 + k(e-1)/n$.

Therefore $\displaystyle\int_1^e \ln\left(x^3\right)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + k\frac{e-1}{n}\right)\frac{e-1}{n}$.

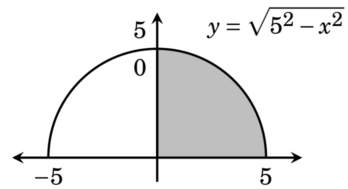
3. Using Definition 41.1, write out the integral and limit for the area under the curve $y = \sin(x)$ between $x = 0$ and $x = \pi$.

The integral is $\displaystyle\int_0^\pi \sin(x)\,dx$. Here $\Delta x = (\pi-0)/n = \pi/n$ and $x_k = 0 + k\Delta x = k\pi/n$.

Therefore $\displaystyle\int_0^\pi \sin(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{k\pi}{n}\right)\frac{\pi}{n}$.

5. Find $\int_0^5 \sqrt{25-x^2}\,dx$ by considering area.

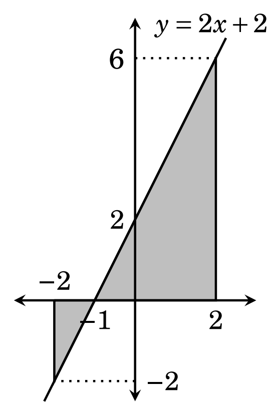
The graph of $y = \sqrt{25-x^2} = \sqrt{5^2-x^2}$ is the upper half of a circle of radius 5 centered at the origin. From the picture, the integral is one fourth of the area of a circle of radius 5, so $\int_0^5 \sqrt{25-x^2}\,dx = \pi 5^2/4 = 25\pi/4$.



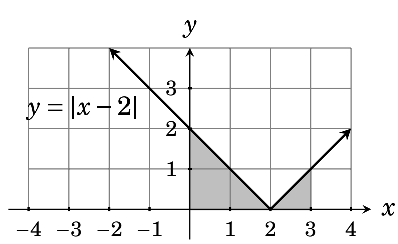
7. Find $\int_{-2}^{2} 2x+2\,dx$ by considering area.

Draw the graph of $y = 2x+2$, which is a line with slope 2 and y-intercept 2. The relevant region consists of two triangles, one below the x-axis, of area $\frac{1}{2}\cdot 1\cdot 2 = 1$, and the other above the x-axis, of area $\frac{1}{2}\cdot 3\cdot 6 = 9$. Then

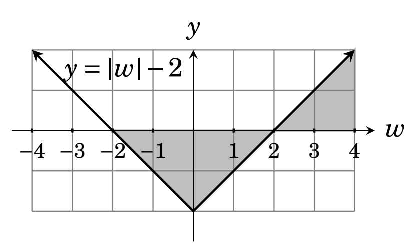
$$\int_{-2}^{2} 2x+2\,dx = A_{\text{up}} - A_{\text{down}} = 9 - 1 = 8.$$



9. $\int_0^3 |x-2|\,dx = 2.5$ (graph below).



11. $\int_{-2}^{4} (|w|-2)\,dw = A_{\text{up}} - A_{\text{down}} = 2 - 4 = -2$



13. Suppose f is a function for which $\int_1^5 f(x)dx = 3$ and $\int_1^7 f(x)dx = -6$. Find $\int_5^7 f(x)dx$.
By Property 6, $\int_1^7 f(x)dx = \int_1^5 f(x)dx + \int_5^7 f(x)dx$. This yields $-6 = 3 + \int_5^7 f(x)dx$, so $\int_5^7 f(x)dx = -9$.

15. Suppose f is a function for which $\int_2^5 f(x)dx = 4$ and $\int_2^8 f(x)dx = 9$. Find $\int_8^5 7f(x)dx$.
By Property 6, $\int_2^8 f(x)dx = \int_2^5 f(x)dx + \int_5^8 f(x)dx$. Applying Property 3 to this, we get $\int_2^8 f(x)dx = \int_2^5 f(x)dx - \int_8^5 f(x)dx$. Inserting the given values into this yields $9 = 4 - \int_8^5 f(x)dx$, so $\int_8^5 f(x)dx = -5$. Then $\int_8^5 7f(x)dx7 = 7\int_8^5 f(x)dx = 7\cdot(-5) = -35$.

17. Write $\int_0^2 f(x)\,dx - \int_{-2}^2 f(x)\,dx - \int_{-4}^{-2} f(x)\,dx$ as a **single integral** of the form $\int_a^b f(x)\,dx$.
Applying Property 6 to the middle term, this becomes
$\int_0^2 f(x)\,dx - \left(\int_{-2}^0 f(x)\,dx + \int_0^2 f(x)\,dx\right) - \int_{-4}^{-2} f(x)\,dx = -\int_{-2}^0 f(x)\,dx - \int_{-4}^{-2} f(x)\,dx =$
$-\left(\int_{-4}^{-2} f(x)\,dx + \int_{-2}^0 f(x)\,dx\right) = -\int_{-4}^0 f(x)\,dx = \int_0^{-4} f(x)\,dx.$

19. Because the graph of f is composed of straight lines on $[-4,0]$ it is easy see that there are six square units of area under the graph of f between -4 and 0. Thus $\int_{-4}^0 f(x)dx = 6$. Integral property 6 implies $\int_{-4}^4 f(x)dx = \int_{-4}^0 f(x)dx + \int_0^4 f(x)dx$. From this, $17.8 = 6 + \int_0^4 f(x)dx$, and therefore $\int_0^4 f(x)dx = 11.8$.

21. For a positive integer n, let $\Delta x = 1/n$, and for each integer k (where $1 \le k \le n$), let $x_k = k\Delta x$. Consider the limit $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} x_k\sqrt{x_k^2+1}\,\Delta x$. Write a definite integral in the form $\int_a^b f(x)dx$ that equals this limit.

As k goes from 0 to n, the numbers $x_k = k\Delta x$ go from 0 to $n\Delta x = n\frac{1}{n} = 1$. That is, they progress, equally spaced, along the interval $[0,1]$. Definition 41.1 implies
$$\lim_{n\to\infty}\sum_{k=1}^{n} x_k\sqrt{x_k^2+1}\,\Delta x = \int_0^2 x\sqrt{x+1}\,dx$$

23. Write the limit $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}(-1+2k/n)\cdot\cos\left((-1+2k/n)^2\right)\dfrac{2}{n}$ as a definite integral.

As k goes from 1 to n, the numbers $-1+2k/n$ that appear in the above expression go from $-1+2/n$ to $-1+2n/n = 1$, which indicates the interval $[-1,1]$, for which $\Delta x = (1-(-1)/n = 2/n$ and $x_k = -1+2k/n$. Then:
$$\lim_{n\to\infty}\sum_{k=1}^{n}(-1+2k/n)\cdot\cos\left((-1+2k/n)^2\right)\frac{2}{n} = \lim_{n\to\infty}\sum_{k=1}^{n} x_k\cdot\cos\left((x_k)^2\right)\Delta x = \int_{-1}^{1} x\cos\left(x^2\right)\,dx.$$