CHAPTER 39

Differentials and Differential Equations

Often in applications, information about the derivative of a function is known, and the function must be found. We now study such problems.

39.1 Initial Value Problems

When we find the indefinite integral of a function $f(x)$, we don't get a single function, but rather infinitely many different functions $F(x)+C$, one function for each different value of C:

$$\int f(x)\,dx \;=\; F(x)+C.$$

But in some applications, extra information is available that gives a value for C. For example, suppose we happen to know that, say, $F(2)+C=5$. Then $C=5-F(2)$. This kind of problem is called an **initial value problem**.

Example 39.1 Suppose $g(x)$ is a function for which $g'(x)=x+\dfrac{1}{1+x^2}$ and $g(1)=7$. Find $g(x)$.

Solution Because $g'(x)=x+\frac{1}{1+x^2}$, $g(x)$ is an antiderivative of $x+\frac{1}{1+x^2}$, so

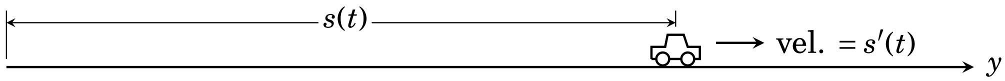
$$g(x) \;=\; \int\left(x+\frac{1}{1+x^2}\right)dx \;=\; \frac{x^2}{2}+\tan^{-1}(x)+C.$$

So $g(x)=\frac{x^2}{2}+\tan^{-1}(x)+C$, but we still don't know g exactly because we don't know C. But we do know $g(1)=7$, and this is enough to get a value of C:

$$\begin{aligned} g(1) &= 7 \\ \frac{1^2}{2}+\tan^{-1}(1)+C &= 7 \\ \frac{1}{2}+\frac{\pi}{4}+C &= 7 \\ C &= 7-\frac{1}{2}-\frac{\pi}{4} \;=\; \frac{26-\pi}{4} \end{aligned}$$

Answer Having found C, we know g exactly: $g(x)=\dfrac{x^2}{2}+\tan^{-1}(x)+\dfrac{26-\pi}{4}$.✍

Initial value problems occur often in the context of motion. Recall that if an object moving on a straight line has position $s(t)$ at time t, then its velocity at time t is $v(t) = s'(t)$.



Having studied antiderivatives, we see this in a new light. Since velocity is the derivative of position, *position is an antiderivative of velocity*, that is,

$$s(t) = \int v(t)\,dt.$$

Also, as acceleration is the derivative of velocity, then *velocity is an antiderivative of acceleration:*

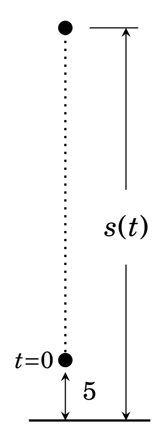
$$v(t) = \int a(t)\,dt.$$

In summary, the following formulas apply to motion on a straight line.

$$\begin{array}{rrcccl} \text{Position at time } t: & s(t) & = & \int v(t)\,dt & & \\ \text{Velocity at time } t: & v(t) & = & s'(t) & = & \int a(t)\,dt \\ \text{Acceleration at time } t: & a(t) & = & v'(t) & & \end{array}$$

So if you know velocity, you can get position as the integral of velocity. If you know acceleration, then you can get velocity as the integral of acceleration. Taking the integrals may introduce some constants, but often initial value conditions apply and yield exact expressions for these functions.

Example 39.2 A ball tossed straight up has a constant acceleration of -32 feet per second per second. At time $t=0$ its velocity is 20 ft/sec, and it is 5 feet high. Find the position function $s(t)$ giving the ball's height at time t.



Solution We can obtain velocity from acceleration as $v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C$. So $v(t) = -32t + C$. To find C, use the fact that velocity at time 0 is $v(0) = 20$ ft/sec. Thus $20 = v(0) = -32 \cdot 0 + C$, so $20 = C$, and $\boxed{v(t) = -32t + 20.}$

We seek the position function, and it is an antiderivative of velocity, so $s(t) = \int v(t)\,dt = \int(-32t + 20)\,dt = -16t^2 + 20t + C$, and so $s(t) = -16t^2 + 20t + C$. We just need to find C. Because the ball's height at time $t = 0$ is 5 feet, we have $5 = s(0) = -16 \cdot 0^2 + 20 \cdot 0 + C$, which gives $C = 5$. Thus $s(t) = -16t^2 + 20t + 5$.

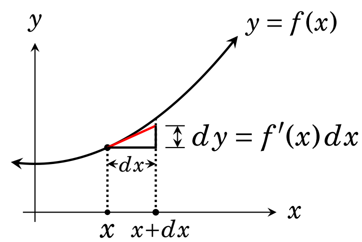
Answer: The position function is $\boxed{s(t) = -16t^2 + 20t + 5}$. ✍

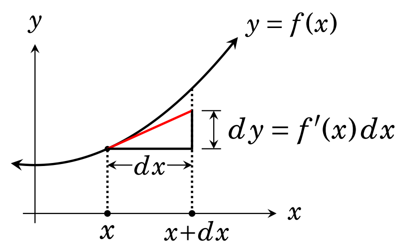
39.2 Differentials

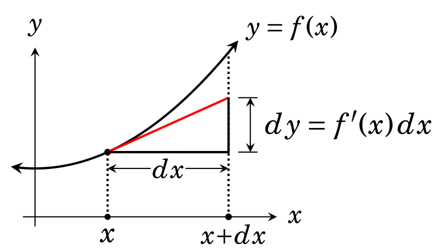
Thus far, certain expressions dx and dy have appeared in our notation. For instance, an indefinite integral $\int f(x)\,dx$ ends with a dx. Also, $\frac{dy}{dx}$ stands for the derivative of a function $y = f(x)$, that is, $\frac{dy}{dx} = f'(x)$. Until now, we have viewed the expression $\frac{dy}{dx}$ as a single notational package that stands for $f'(x)$. We have not assigned any formal meaning to the numerator dy and the denominator dx. We do so now. These terms have special meanings, and they are called *differentials*.

We define dx and dy to be *variables*. Although they can potentially have any value, we usually regard dx and dy as standing for small positive numbers. We interpret them as follows.

Given a value x in the domain of a function f, the variable dx is viewed as a quantity that is added to x. So if dx is positive, then $x+dx$ is to the right of x. (See the diagrams on the right.)







The variable dy is defined to be the product of $f'(x)$ and dx, namely

$$dy = f'(x) \cdot dx.$$

Notice that, when dx and dy are defined this way, then dy divided by dx is

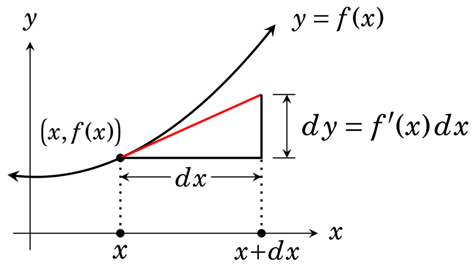
$$\frac{dy}{dx} = \frac{f'(x) \cdot dx}{dx} = f'(x).$$

In other words, under this setup the quotient $\frac{dy}{dx}$ really does equal $f'(x)$.

Geometrically, the variables dy and dx can be viewed as the side lengths of a right triangle obtained by moving a horizontal distance of dx from the point $\big(x, f(x)\big)$, then a vertical distance of dy, and then back to $\big(x, f(x)\big)$. This is shown above for several values of dx. Regardless of the value dx, the hypotenuse has slope $\frac{\text{rise}}{\text{run}} = \frac{dy}{dx} = f'(x)$, so the hypotenuse is tangent to $y = f(x)$ at $\big(x, f(x)\big)$.

Variables dx and dy have a special name. They are called *differentials*.

Definition 39.1 Consider a differentiable function $f(x)$, and a number x in its domain. Under this circumstance, the **differentials** dx and dy are two variables related by the equation $dy = f'(x)\,dx$.

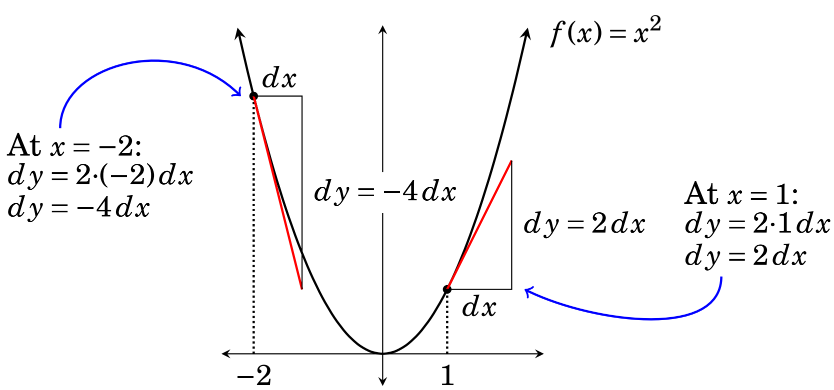


Geometrically the differentials dx and dy have the following interpretation: Starting at the point $\big(x, f(x)\big)$, move along the tangent line to $y = f(x)$. If the run is dx, then the rise will be dy.

Notice that the equation $dy = f'(x)dx$ contains *three* variables: dy, dx *and* x. As dx increases, the triangle grows, and its vertical height dy increases by a factor of $f'(x)$. This can happen at different points x, so the growth factor $f'(x)$ depends on x. The next example illustrates this.

Example 39.3 Consider the function $f(x) = x^2$. We will now examine the differentials dx and dy at two different values of x. For this particular function f, the equation $dy = f'(x)dx$ is $\boxed{dy = 2x\,dx.}$

Say $x = 1$. Then $dy = 2 \cdot 1\,dx$, or $dy = 2\,dx$. This means that at $x = 1$, dy is always twice what dx is. (See the diagram below.) Moving along the tangent line from $(1, f(1)) = (1, 1)$, your run will be dx and your rise will be $dy = 2\,dx$.



But if (say) $x = -2$, then $dy = 2 \cdot (-2)\,dx$, or $dy = -4\,dx$. Starting at $(-2, f(-2))$ and tracing the tangent line, for a run of dx, the rise is $dy = -4\,dx$. ✍

In working with differentials, one must be very sensitive to the variables used. For example, the function $s = g(t)$ has differentials ds and dt, with $ds = g'(t)\,dt$.

One reason that differentials are useful is that they facilitate a certain "separation of variables." They allow an equation like

$$\frac{dy}{dx} = f'(x)$$

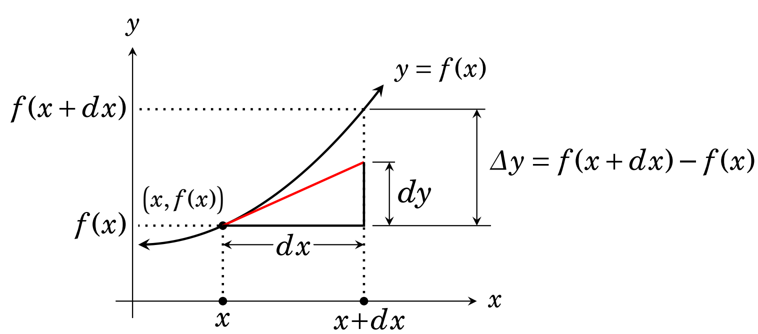
to be written in the equivalent form

$$dy = f'(x)\,dx,$$

with the y's on one side and the x's on the other. This will be extremely useful in the next section.

Differentials have an interesting and instructive history. The derivative notation $\frac{dy}{dx}$ goes back to Gottfried Leibniz (1646–1716), who (along with Issac Newton) is credited with the invention of calculus. In Leibniz and Newton's time the concept of a limit had not even been invented, so they arrived at derivatives differently than we do today.

Leibniz regarded dx as a very small increment to x, as illustrated below. Let's denote the resulting change in y as $\Delta y = f(x+dx) - f(x)$ (see below).



From our modern perspective, the derivative $f(x)$ is

$$\begin{aligned} f'(x) &= \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} \\ &= \lim_{dx \to 0} \frac{\Delta y}{dx}. \end{aligned}$$

Consequently

$$f'(x) \approx \frac{\Delta y}{dx}$$

and the approximation becomes better and better the smaller dx is. Leibniz didn't have limits, so he regarded $f'(x)$ as the quotient $\frac{\Delta y}{dx}$ where dx is "infinitely small." For an "infinitely small" dx, he denoted the corresponding 'infinitely small" change Δy as dy. Thus he arrived at

$$f'(x) = \frac{dy}{dx}.$$

Leibniz called the infinitely small quantities dx and dy **infinitesimals**.

Although building calculus on a foundation of "infinitely small quantities" was logically questionable, it did lead to the spectacular theory of what we now call calculus. Today we regard dx and dy not as infinitely small quantities, but as differentials, as defined by Definition 39.1. This has the advantage of giving the notation $\frac{dy}{dx}$ a clear meaning, while avoiding the logical pitfalls of infinitesimals.

39.3 Differential Equations

[This section is yet to be written. The material to be contained in it is not included in VCU's MATH 200.]

Exercises for Chapter 39

1. Suppose $f(x)$ is a function for which $f'(x) = \sqrt{x} + 2$ and $f(4) = 7$. Find $f(x)$.

2. Suppose $f(x)$ is a function for which $f'(x) = \dfrac{1}{x} + 3x$ and $f(1) = 5$. Find $f(x)$.

3. Suppose $f(x)$ is a function for which $f'(x) = 2x + \cos(x)$ and $f(\pi) = 0$. Find $f(x)$.

4. Suppose $f(x)$ is a function for which $f'(x) = \frac{1}{2}\sec(x)\tan(x)$ and $f(0) = 1$. Find $f(x)$.

5. Suppose $f(x)$ is a function for which $f'(x) = \dfrac{1}{x} + \dfrac{1}{x^2} + 1$ and $f(1) = 3$. Find $f(x)$.

6. Suppose $f(x)$ is a function for which $f'(x) = \dfrac{3}{\sqrt[3]{x^2}}$ and $f(-1) = -5$. Find $f(x)$.

7. Suppose $f(x)$ is a function for which $f'(x) = \dfrac{1}{x^3} + x$. The graph of f passes through the point $(2, 11)$. Find $f(x)$.

8. Suppose $f(x)$ is a function for which $f'(x) = 3x^2 + 1$. The graph of f passes through the point $(1, 3)$. Find $f(x)$.

9. Suppose an object moving on a line has velocity function $v(t) = 2t + 3$. Find its position function $s(t)$, given that you happen to know $s(2) = 8$.

10. An object moving on a line has velocity function $v(t) = \frac{3\sqrt{t}}{2} + 3$. Find its position function $s(t)$, given that you happen to know $s(4) = 10$.

11. A ball, tossed straight up, has a constant acceleration of -32 feet per second per second. At time $t = 0$ its velocity is $v(0) = 10$ feet per second, and its position is $s(0) = 6$ feet. Find the position function $s(t)$.

12. A falling object has a velocity of $-32t - 16$ feet per second t seconds after it is dropped. It hits ground 10 seconds after being dropped. From what height was it dropped?

13. An object moving on the number line has velocity $v(t) = 4t^3$ at time t seconds. At time $t = 1$ it is at the point 4 on the line. When is the object at the point 19?

14. An object moving on the number line has velocity $v(t) = 3t^2 + 4$ at time t seconds. It is at the point 2 on the number line the instant its acceleration is 12 units per second per second. Find the position function $s(t)$.

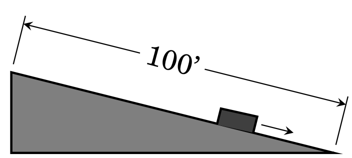
15. A helicopter is rising vertically at a rate of 32 feet per second. At the instant it is 48 feet above the ground, a package is dropped from it. Assuming the acceleration due to gravity is -32 feet per second per second, find the velocity at which the package strikes ground.

16. An object moving on a line has an acceleration function of $a(t) = 12 - 12t^2$. Its position function satisfies $s(0) = 0$ and $s(1) = 6$. Find the position function $s(t)$.

17. A rock, propelled straight down from the top of a bridge over a river at time $t = 0$ seconds has a velocity of $v(t) = -32t - 5$ feet per second at time t. The rock hits water with a velocity of -69 feet per second. How high is the bridge?

18. A bus has stopped to pick up riders, and Richard is running at a rate of 10 feet per second to catch it. When he is 25 feet behind the front door of the bus, it begins to pull away with a constant acceleration of 2 feet per second per second. Will Richard reach the front door of the bus? If so, when will this happen?

19. A block sliding down a 100–foot-long inclined plane has a constant acceleration of 2 feet per second per second. It takes the block five seconds to slide from the top to the bottom. What is its velocity when it reaches the bottom?



20. A freight train, moving with a constant acceleration on a straight track, travels 20 miles in half an hour. At the beginning of the half hour period its velocity is 10 miles per hour. What is its velocity at the end of the half hour period?

21. A car has a velocity of 10 meters per second when the driver sees a stopped car 13 meters away and immediately applies the breaks. The car then de-accelerates at a constant rate of 5 meters per second per second (that is, its acceleration is -5 meters per second per second). How long does it take the car to stop? Does it stop in time?

Exercise Solutions for Chapter 39

1. Suppose $f(x)$ is a function for which $f'(x) = \sqrt{x} + 2$ and $f(4) = 7$. Find $f(x)$.

Solution First, $f(x) = \int (\sqrt{x}+2)\, dx = \frac{2}{3}\sqrt{x}^3 + 2x + C$. So $7 = f(4) = \frac{2}{3}\sqrt{4}^3 + 2\cdot 4 + C = \frac{40}{3} + C$. Thus $C = 7 - \frac{40}{3} = -\frac{19}{3}$, so $f(x) = \frac{2}{3}\sqrt{x}^3 + 2x - \frac{19}{3}$.

3. Suppose $f(x)$ is a function for which $f'(x) = 2x + \cos(x)$ and $f(\pi) = 0$. Find $f(x)$.

Solution First, $f(x) = \int (2x + \cos(x))\, dx = x^2 + \sin(x) + C$. Now $0 = f(\pi) = \pi^2 + \sin(\pi) + C = \pi^2 + C$, so $C = -\pi^2$. Therefore $f(x) = x^2 + \sin(x) - \pi^2$.

5. Suppose $f(x)$ is a function for which $f'(x) = \dfrac{1}{x} + \dfrac{1}{x^2} + 1$ and $f(1) = 3$. Find $f(x)$.

Solution First, $f(x) = \int \left(\frac{1}{x} + \frac{1}{x^2} + 1\right) dx = \ln|x| - \frac{1}{x} + x + C$. Then $3 = f(1) = \ln|1| - \frac{1}{1} + 1 + C = C$, so $C = 3$. Therefore $f(x) = \ln|x| - \frac{1}{x} + x + 3$.

7. Suppose $f(x)$ is a function for which $f'(x) = \dfrac{1}{x^3} + x$. The graph of f passes through the point $(2, 11)$. Find $f(x)$.

Solution First, $f(x) = \int\left(\frac{1}{x^3} + x\right) dx = -\frac{1}{2x^2} + \frac{x^2}{2} + C$. Now $11 = f(2) = -\frac{1}{2\cdot 2^2} + \frac{2^2}{2} + C = -\frac{1}{8} + 2 + C = \frac{15}{8} + C$, so $C = 11 - \frac{15}{8} = \frac{73}{8}$. Therefore $f(x) = -\frac{1}{2x^2} + \frac{x^2}{2} + \frac{73}{8}$.

9. Suppose an object moving on a line has velocity function $v(t) = 2t + 3$. Find its position function $s(t)$, given that you happen to know $s(2) = 8$.

Solution The position function is $s(t) = \int v(t)\, dt = \int (2t+3)\, dt = t^2 + 3t + C$. Then $8 = s(2) = 2^2 + 3\cdot 2 + C = 10 + C$. Thus $C = 8 - 10 = -2$, so $s(t) = t^2 + 3t - 2$.

11. A ball, tossed straight up, has a constant acceleration of -32 feet per second per second. At time $t = 0$ its velocity is $v(0) = 10$ feet per second, and its position is $s(0) = 6$ feet. Find the position function $s(t)$.

Solution First, $v(t) = \int a(t)\, dt = \int -32\, dt = -32t + C$. So $v(t) = -32t + C$. To find C, use the fact that velocity at time 0 is $v(0) = 10$ ft/sec. Thus $10 = v(0) = -32\cdot 0 + C$, so $10 = C$, and $v(t) = -32t + 10$.

Next, $s(t) = \int v(t)\, dt = \int(-32t + 10)\, dt = -16t^2 + 10t + C$, and so $s(t) = -16t^2 + 10t + C$. We just need to find C. Because the ball's height at time $t = 0$ is 6 feet, we have $6 = s(0) = -16\cdot 0^2 + 10\cdot 0 + C$, which gives $C = 6$. Thus $s(t) = -16t^2 + 10t + 6$.

13. An object moving on the number line has velocity $v(t) = 4t^3$ at time t seconds. At time $t = 1$ it is at the point 4 on the line. When is the object at the point 19?

Solution The position function is $s(t) = \int v(t)\, dt = \int 4t^3\, dt = t^4 + C$. To find C, we have $4 = s(1) = 1^4 + C$. Thus $C = 4 - 1 = 3$, so $s(t) = t^4 + 3$. To find when the object is at the point 19, we must solve the equation $s(t) = 19$, that is, $t^4 + 3 = 19$. Then $t^4 = 16$ so $t = 2$. Thus at time $t = 2$ seconds, the object is at the point 19.

15. A helicopter is rising vertically at a rate of 32 feet per second. At the instant it is 48 feet above the ground, a package is dropped from it. Assuming the acceleration due to gravity is −32 feet per second per second, find the velocity at which the package strikes ground.

Solution Here is a strategy: The acceleration $a(t)$ of the package is given. Working backwards, we will try to use the given information to derive the package's velocity function $v(t)$ and position function $s(t)$ (i.e., its height above ground at time t). Then to find the instant it strikes ground we will solve $s(t) = 0$. We will then plug this time value into $v(t)$ to get the velocity on impact.

Start the clock (time $t = 0$) the moment the helicopter is 48 feet high, which is also the instant that the package is dropped. At this instant the package's velocity is the velocity of the helicopter, which is 32 feet per second (straight up). So letting $v(t)$ be the package's velocity at time t, we have $v(0) = 32$.

The package's acceleration is given as $a(t) = -32$, so its velocity is $v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C$ feet per second, that is, $v(t) = -32t + C$. To find C, recall that $v(0) = 32$, hence $32 = v(0) = -32 \cdot 0 + C$, so $C = 32$. Thus $v(t) = -32t + 32$.

The package's height at time t is $s(t) = \int v(t)\,dt = \int(-32t + 32)\,dt = -16t^2 + 32t + C$. To find C, recall that the package is 48 feet high at time $t = 0$, so $48 = s(0) = -16 \cdot 0^2 + 32 \cdot 0 + C$, and hence $C = 48$. Therefore $s(t) = -16t^2 + 32t + 48$.

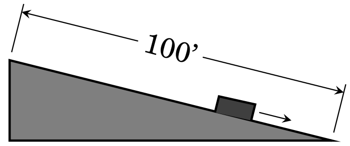
The package hits ground when it height is 0, that is when $s(t) = 0$, or $-16t^2 + 32t + 48 = 0$. Dividing both sides, by −16 gives $t^2 - 2t - 3 = 0$, or $(t+1)(t-3) = 0$. This gives two time values, $t = -1$ and $t = 3$. The first is not in our interval of consideration, so we conclude that the package strikes ground at time $t = 3$ seconds. At this instant its velocity is $v(3) = -32 \cdot 3 + 32 = \boxed{-64 \text{ feet per second.}}$

17. A rock, propelled straight down from the top of a bridge over a river at time $t = 0$ seconds has a velocity of $v(t) = -32t - 5$ feet per second at time t. The rock hits water with a velocity of −69 feet per second. How high is the bridge?

Solution To find the time that the rock hits the water, we just have to solve $v(t) = -69$, that is, $-32t - 5 = -69$. This reduces to $32t = 64$, giving $t = 2$. So the rock hits the water at time $t = 2$ seconds.

To find the bridge's height, let $s(t)$ be height of the rock at time t. If we can find $s(t)$, the height of the bridge will be $s(0)$ (the rock's height when it's at the top of the bridge). Now, $s(t) = \int v(t)\,dt = \int(-32t - 5)\,dt = -16t^2 - 5t + C$. We know $s(5) = 0$ since the rock has reaches the water (has height 0) at time $t = 5$. Then $0 = s(5) = -16(5^2) - 5 \cdot 5 + C = -425 + C$. Thus $C = 425$, and hence $s(t) = -16t^2 - 5t + 425$. So the height of the bridge is $s(0) = -16 \cdot 0^2 - 5 \cdot 0 + 425 = \boxed{425 \text{ feet.}}$

19. A block sliding down a 100–foot-long inclined plane has a constant acceleration of 2 feet per second per second. It takes the block five seconds to slide from the top to the bottom. What is its velocity when it reaches the bottom?



Solution Here is our strategy: Let $s(t)$ be the block's distance from the top of the ramp at time t. Start the clock at time $t = 0$ when the block is at the top. Thus $s(0) = 0$. The block is at the bottom of the ramp at time $t = 5$, so $s(5) = 100$. We are asked for the velocity at this time, so we will construct a velocity function $v(t)$, and then our answer will be $v(5)$.

The block's acceleration is given as $a(t) = 2$, so the velocity is $v(t) = \int a(t)\,dt = \int 2\,dt = 2t + C$. We are not given an initial value for velocity that would allow us to solve for C, so let's integrate again to get $s(t)$ and see if what we already know about $s(t)$ will help solve for C.

Now, $s(t) = \int v(t)\,dt = \int (2t + C)\,dt = t^2 + Ct + K$, where K is a new constant. So $s(t) = t^2 + Ct + K$. We have $0 = s(0) = 0^2 + C \cdot 0 + K$, so $K = 0$, and hence $s(t) = t^2 + Ct$. Now use the fact $s(5) = 100$ to get $100 = s(5) = 5^2 + C \cdot 5$. Then $C \cdot 5 = 75$, so $C = 15$.

Now that we know C we can finally get the velocity function: $v(t) = 2t + 15$. Thus the velocity at the bottom of the ramp is $v(5) = 2 \cdot 5 + 15 = \boxed{\text{25 feet per second.}}$

20. A car has a velocity of 10 meters per second when the driver sees a stopped car 13 meters away and immediately applies the breaks. The car then de-accelerates at a constant rate of 5 meters per second per second (that is, its acceleration is −5 meters per second per second). At this rate, how long will it take the car to stop? Does it stop in time?

Say the car is moving on the number line so that the stopped car is at the point 13, and the breaks are applied when the car is at the point 0, at time $t = 0$. Let $s(t)$ be its position function, so that $s(0) = 0$. We are also given the information $v(0) = 10$.

The acceleration is given as $a(t) = -5$, so $v(t) = \int a(t)dt = \int -5\,dt = -5t + C$. So $v(t) = -5t + C$. To find C note that $10 = v(0) = -5 \cdot 0 + C$, so $C = 10$ and hence $v(t) = -5t + 10$. From this you can see that the car has zero velocity (i.e. come to a stop) at time $t = 2$ seconds.

To answer the second question we will find $s(t)$. If $s(2) < 13$, then the car has stopped in time. We have $s(t) = \int v(t)dt = \int (-5t + 10)dt = -\frac{5}{2}t^2 + 10t + C$. The condition $0 = s(0) = -\frac{5}{2} \cdot 0^2 + 10 \cdot + C$ yields $C = 0$, so $s(t) = -\frac{5}{2}t^2 + 10t$. The car stops at time $t = 2$, and at this time it is at the point $s(2) = -\frac{5}{2} \cdot 2^2 + 10 \cdot 2 = 10$. It is still 3 meters away from the stopped car, so it stopped in time.