CHAPTER 37

L'Hôpital's Rule

As you go further in mathematics, you will encounter a great variety of limits, and you will need to know how to evaluate them. In this chapter we will see that differentiation can be applied to the problem of computing limits. The new technique that we will explore is called *L'Hôpital's rule*, and the types of limits that we will apply it to are called *indeterminate forms*.

37.1 The Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

The most significant limits in calculus have form $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$, where both the numerator $f(x)$ and the denominator $g(x)$ approach 0. For example, the slope of the tangent to $y = \sqrt{x}$ at $x = 4$ is given by the limit $\lim\limits_{x\to 4} \frac{\sqrt{x}-\sqrt{4}}{x-4} = \frac{1}{4}$. In this limit both the numerator and denominator approach 0. In such a circumstance we say that the limit has the **indeterminate form** $\frac{0}{0}$. Here are three familiar limits with this form.

$$\left.\begin{aligned} &\lim_{x\to 4} \frac{\sqrt{x}-\sqrt{4}}{x-4} = \frac{1}{4} \\ &\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \\ &\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0 \end{aligned}\right\} \text{ indeterminate forms } \tfrac{0}{0}$$

Again, a limit has *indeterminate form* $\frac{0}{0}$ if its structure is $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$, where $\lim\limits_{x\to c} f(x) = 0 = \lim\limits_{x\to c} g(x)$. By itself, $\frac{0}{0}$ is not defined. But limits having this form can exist and can equal different values, as seen in the examples above.

Another indeterminate form that we have seen is $\frac{\infty}{\infty}$. Consider these:

$$\left.\begin{aligned} &\lim_{x\to\infty} \frac{3x^2-4x-5}{2x^2+x+11} = \frac{3}{2} \\ &\lim_{x\to 0} \frac{1+1/x}{2+1/x} = 1 \end{aligned}\right\} \text{ indeterminate forms } \tfrac{\infty}{\infty}$$

In these limits, the numerator and denominator both approach ∞. By itself, $\frac{\infty}{\infty}$ makes no sense, but limits having this form may well make sense.

In Part 2 of this text we developed some algebraic techniques for dealing with limits of indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, although we did not call them "indeterminate forms" at the time. But our techniques (such as canceling denominators, etc.) did not always work. We now introduce a powerful technique, one that works on limits that our old techniques couldn't handle.

Fact 37.1 L'Hôpital's Rule
If $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$ exists and has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$.

L'Hôpital's rule says that if you are ever trying to find $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$, and this limit has an indeterminate form of either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{\frac{d}{dx}\big[f(x)\big]}{\frac{d}{dx}\big[g(x)\big]}.$$

Because differentiating a function can simplify it (powers reduce, constants disappear, etc.), L'Hôpital's rule can change a hard limit into an easy one. To illustrate, let's apply L'Hôpital's rule to the limits on the previous page.

$$\lim_{x\to 4}\frac{\sqrt{x}-\sqrt{4}}{x-4} = \lim_{x\to 4}\frac{\frac{1}{2\sqrt{x}}-0}{1-0} = \lim_{x\to 4}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = \cos(0) = 1$$

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} = \lim_{x\to 0}\frac{0+\sin(x)}{1} = \frac{0+0}{1} = 0$$

$$\lim_{x\to 0}\frac{1+1/x}{2+1/x} = \lim_{x\to 0}\frac{0-1/x^2}{0-1/x^2} = \lim_{x\to 0}\frac{-1/x^2}{-1/x^2} = 1$$

Sometimes after applying L'Hôpital's rule you will get another limit of indeterminate form. In such a case you can apply L'Hôpital's rule again. For example $\lim\limits_{x\to\infty}\frac{3x^2-4x-5}{2x^2+x+11}$ from the previous page has form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{3x^2-4x-5}{2x^2+x+11} = \lim_{x\to\infty}\frac{6x-4}{4x+1} \qquad \text{(L'Hôpital's rule)}$$

This resulting limit still has form $\frac{\infty}{\infty}$, so we apply L'Hôpital's rule to *it*.

$$= \lim_{x\to\infty}\frac{6-0}{4+0} = \frac{6}{4} = \frac{3}{2} \qquad \text{(L'Hôpital's rule)}$$

Our statement of L'Hôpital's rule (Fact 37.1) assumes that that $f'(x)$ and $g'(x)$ actually exist on an interval containing c (except possibly at $x=c$) so that the limits makes sense. You don't have to worry about this detail in the exercises.

A proof of L'Hôpital's rule is best left to more advanced calculus courses. But here is a quick informal justification of it. Suppose $\lim_{x\to c}\frac{f(x)}{g(x)}$ has form $\frac{0}{0}$. This means that $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=0$. Assume that f, g, f' and g' are continuous at c. (This is a big assumption. They need not be continuous for L'Hôpital's rule to work, but this is just an informal justification.) In particular, by continuity, $0=\lim_{x\to c} f(x)=f(c)$ and $0=\lim_{x\to c} g(x)=g(c)$. Note that

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}=\lim_{x\to c}\frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(c)}{x-c}}=\frac{f'(c)}{g'(c)}=\frac{\lim_{x\to c}f'(x)}{\lim_{x\to c}g'(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Thus $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$. This is L'Hôpital's rule.

For the remainder of this section we'll work examples using L'Hôpital's rule on limits of forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Example 37.1 Find $\displaystyle\lim_{x\to 0}\frac{e^x-1}{\sin(x)}$.

The numerator approaches $e^0-1=0$, and the denominator approaches $\sin(0)=0$. So the limit has indeterminate form $\frac{0}{0}$, and we can use L'Hôpital's rule. Doing so, $\displaystyle\lim_{x\to 0}\frac{e^x-1}{\sin(x)}=\lim_{x\to 0}\frac{e^x}{\cos(x)}=\frac{e^0}{\cos(0)}=\frac{1}{1}=\boxed{1.}$

Example 37.2 Find $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}$.

This limit has form $\frac{0}{0}$. Clearly we can find it by factoring the numerator and cancelling. We can also use L'Hôpital's rule: $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{2x}{1}=\boxed{2.}$

Example 37.3 Find $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{1-\cos(2x)}$.

In this limit, the numerator approaches $e^0-e^0-2\cdot 0=0$, and the denominator approaches $1-\cos(0)=0$. So the limit has form $\frac{0}{0}$. Applying L'Hôpital's rule, $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{1-\cos(2x)}=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{-2\sin(2x)}$. This new limit also has form $\frac{0}{0}$, so apply L'Hôpital's rule to *it*: $\displaystyle\lim_{x\to 0}\frac{e^x+e^{-x}-2}{-2\sin(2x)}=\lim_{x\to 0}\frac{e^x-e^{-x}}{-4\cos(2x)}=\frac{e^0-e^{-0}}{-4\cos(0)}=\frac{1-1}{-4}=\boxed{0.}$

Example 37.4 Find $\lim_{x\to 0^+} \frac{\ln(x)}{\cot(x)}$.

Since $\lim_{x\to 0^+} \ln(x) = \infty$ and $\lim_{x\to 0^+} \cot(x) = \infty$, this has form $\frac{\infty}{\infty}$, and we can apply L'Hôpital's rule to it:

$$\lim_{x\to 0^+} \frac{\ln(x)}{\cot(x)} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2(x)}.$$

The resulting limit still has form $\frac{\infty}{\infty}$. But before applying L'Hôpital's rule again, notice that doing so would make the numerator more complex. In such cases try simplifying the limit before applying L'Hôpital's rule:

$$= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x}.$$

We have arrived a limit that has form $\frac{0}{0}$. Applying L'Hôpital's rule again:

$$= \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{1} = \boxed{0.}$$ ✍

L'Hôpital's rule also works if $x \to \pm\infty$, as in the next example.

Example 37.5 Find $\lim_{x\to\infty} \frac{e^x-1}{e^x+1}$.

This limit has form $\frac{\infty}{\infty}$, so L'Hôpital's applies: $\lim_{x\to\infty} \frac{e^x-1}{e^x+1} = \lim_{x\to\infty} \frac{e^x}{e^x} = \boxed{1.}$ ✍

The next example illustrates that L'Hôpital's rule handles infinite limits.

Example 37.6 Find $\lim_{x\to 0} \frac{\sin(x)}{x^3}$.

This has form $\frac{0}{0}$, so L'Hôpital applies: $\lim_{x\to 0} \frac{\sin(x)}{x^3} = \lim_{x\to 0} \frac{\cos(x)}{3x^2} = \boxed{\infty.}$ ✍

Be careful not to use L'Hôpital's when it does not apply. Consider, for example, $\lim_{x\to 2} \frac{x-2}{x^2-3}$. This has **neither** form $\frac{0}{0}$ **nor** form $\frac{\infty}{\infty}$, so L'Hôpital's rule does not apply. In fact, $\lim_{x\to 2} \frac{x-2}{x^2-3} = \frac{2-2}{2^2-3} = 0$. If we erroneously used L'Hôpital's rule on it, we'd get the wrong answer.

Likewise $\lim_{x\to 0} \frac{x+1}{x^2}$ has "form $\frac{1}{0}$," so we don't use L'Hôpital's rule here. We learned how to deal with this type of limit in Chapter 12. Since the numerator approaches 1 and the denominator approaches 0 (and is positive), $\lim_{x\to 0} \frac{x+1}{x^2} = \infty$.

37.2 The Indeterminate Forms $\mathbf{0}\cdot\infty$ and $\infty-\infty$

As it turns out, $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are just the first two of a total of seven indeterminate forms. This section investigates the next two: $0\cdot\infty$ and $\infty-\infty$.

Let's first investigate $0\cdot\infty$.

The expression $0\cdot\infty$ is totally ambiguous. Zero times anything should be zero, but shouldn't infinity times anything be infinity? The expression $0\cdot\infty$ seems to be a battle between two opposite entities.

But some limits have this form. We say that a limit $\lim_{x\to c} f(x)g(x)$ has indeterminate form $0\cdot\infty$ if $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=\infty$. Here are two simple examples to show that such limits can exist and equal different values.

$$\left.\begin{aligned} \lim_{x\to 0} x^2\cdot\frac{1}{x^2} &= 1 \\ \lim_{x\to 0} x^2\cdot\frac{5}{x^2} &= 5 \end{aligned}\right\}\ \text{indeterminate forms } 0\cdot\infty$$

These have intentionally obvious answers. Others are not so simple:

$$\left.\begin{aligned} \lim_{x\to 0} x^4\cdot\ln|x| &= ? \\ \lim_{x\to \pi^-} (x-\pi)\tan(x/2) &= ? \end{aligned}\right\}\ \text{indeterminate forms } 0\cdot\infty$$

There is a simple trick for handling this. We can convert the form $0\cdot\infty$ to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as follows. Suppose $\lim_{x\to c} f(x)g(x)$ has indeterminate form $0\cdot\infty$, where $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=\infty$. Then

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \frac{f(x)}{\frac{1}{g(x)}} \quad \text{has form } \tfrac{0}{0},$$

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \frac{g(x)}{\frac{1}{f(x)}} \quad \text{has form } \tfrac{\infty}{\infty}.$$

In other words, whenever $\lim_{x\to c} f(x)g(x)$ has indeterminate form $0\cdot\infty$, bring the reciprocal of either $g(x)$ or $f(x)$ to the denominator. You then have form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and L'Hôpital's rule applies.

For example, take $\lim_{x\to\infty} e^{-x}x$. Since $\lim_{x\to\infty} e^{-x}=0$ and $\lim_{x\to\infty} x=\infty$, this limit has form $0\cdot\infty$. Write it as $\lim_{x\to\infty} e^{-x}x = \lim_{x\to\infty} \frac{x}{\frac{1}{e^{-x}}} = \lim_{x\to\infty} \frac{x}{e^x}$, which has form $\frac{\infty}{\infty}$.

Applying L'Hôpital's rule, $\lim_{x\to\infty} e^{-x}x = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \boxed{0.}$

Example 37.7 Find $\lim_{x\to 0} x^4 \ln|x|$.

Since $\lim_{x\to 0} x^4 = 0$ and $\lim_{x\to 0} \ln|x| = -\infty$, this limit has form $0\cdot\infty$. Work it as

$$\lim_{x\to 0} x^4 \ln|x| = \lim_{x\to 0} \frac{\ln|x|}{\frac{1}{x^4}} \qquad \text{(this now has form } \tfrac{\infty}{\infty}\text{)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-4}{x^5}} \qquad \text{(apply L'Hôpital's rule)}$$

This limit still has form $\frac{\infty}{\infty}$, and applying L'Hôpital's rule a second time would only make it more complicated. Instead, just simplify the fraction:

$$= \lim_{x\to 0} \frac{1}{x}\cdot\frac{x^5}{-4} = \lim_{x\to 0} -\frac{x^4}{4} = \boxed{0}$$

Thus $\lim_{x\to 0} x^4 \ln|x| = 0$. But before moving on, notice that we could have converted this limit to form $\frac{0}{0}$ as

$$\lim_{x\to 0} x^4 \ln|x| = \lim_{x\to 0} \frac{x^4}{\frac{1}{\ln|x|}}. \qquad \text{(this has form } \tfrac{0}{0}\text{)}$$

But then, in applying L'Hôpital's rule, the derivative of $\frac{1}{\ln|x|}$ would be $-\frac{1}{(\ln|x|)^2 x}$, and we'd end up with a mess even worse than the original limit. After some practice with problems like this, you'll learn to look a few steps ahead to gauge which conversion ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) is the better choice. ✍

Example 37.8 Find $\lim_{x\to\pi^-} (x-\pi)\tan(x/2)$.

Since $\lim_{x\to\pi^-} (x-\pi) = 0$ and $\lim_{x\to\pi^-} \tan(x/2) = \infty$, this limit has form $0\cdot\infty$. Do we want to convert it to $\lim_{x\to\pi^-} \frac{x-\pi}{\cot(x/2)}$ (form $\frac{0}{0}$) or $\lim_{x\to\pi^-} \frac{\tan(x/2)}{\frac{1}{x-\pi}}$ (form $\frac{\infty}{\infty}$)? In the first option, the derivative of $x-\pi$ is 1 (simple!) so let's go with it.

$$\lim_{x\to\pi^-} (x-\pi)\tan(x/2) = \lim_{x\to\pi^-} \frac{x-\pi}{\cot(x/2)} \qquad \text{(this now has form } \tfrac{0}{0}\text{)}$$

$$= \lim_{x\to\pi^-} \frac{1}{-\csc^2(x/2)\frac{1}{2}} \qquad \text{(apply L'Hôpital's rule)}$$

$$= \lim_{x\to\pi^-} \frac{-2}{\csc^2(x/2)} = \frac{-2}{\csc^2(\pi/2)} = \boxed{-2.}$$

We've now shown that $\lim_{x\to\pi^-} (x-\pi)\tan(x/2) = -2$. ✍

Next we'll consider the indeterminate form $\infty-\infty$. We say a limit has this form if its structure is $\lim_{x\to c}\big(f(x)-g(x)\big)$, where $\lim_{x\to c} f(x)=\infty$ and $\lim_{x\to c} g(x)=\infty$. Although the knee-jerk impulse may be to say $\infty-\infty=0$, limits with this form can have values other than 0. Here are two transparent examples.

$$\left.\begin{aligned}\lim_{x\to 0}\left(\left(1+\frac{1}{x^2}\right)-\frac{1}{x^2}\right) &= 1\\ \lim_{x\to 0}\left(\left(3+\frac{1}{x^2}\right)-\frac{1}{x^2}\right) &= 3\end{aligned}\right\}\text{ indeterminate forms } \infty-\infty$$

The most common strategy for dealing with such a limit is to attempt to combine the difference $f(x)-g(x)$ into a single term. When this is possible, the resulting expression will involve one of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ or $0\cdot\infty$, with which we are familiar.

Example 37.9 Find $\lim_{x\to\infty}\Big(2\ln(5x+1)-\ln\big(3x^2+x+1\big)\Big)$.

Because $\lim_{x\to\infty} 2\ln(5x+1)=\infty$ and $\lim_{x\to\infty}\ln(3x^2+x+1)=\infty$, this limit has form $\infty-\infty$. We can simplify it with logarithm properties.

$$\begin{aligned}\lim_{x\to\infty}\Big(2\ln(5x+1)-\ln\big(3x^2+x+1\big)\Big) &= \lim_{x\to\infty}\Big(\ln\big((5x+1)^2\big)-\ln\big(3x^2+x+1\big)\Big)\\ &= \lim_{x\to\infty}\ln\left(\frac{(5x+1)^2}{3x^2+x+1}\right)\\ &= \lim_{x\to\infty}\ln\left(\frac{25x^2+10x+1}{3x^2+x+1}\right)\\ &= \ln\left(\lim_{x\to\infty}\frac{25x^2+10x+1}{3x^2+x+1}\right)\end{aligned}$$

The limit inside the ln can be handled either with Chapter 13 techniques or with two applications of L'Hôpital's rule:

$$\begin{aligned}&= \ln\left(\lim_{x\to\infty}\frac{50x+10}{6x+1}\right) && \text{(L'Hôpital's rule)}\\ &= \ln\left(\lim_{x\to\infty}\frac{50}{6}\right) && \text{(L'Hôpital's rule)}\\ &= \boxed{\ln(25/3)}\end{aligned}$$

✍

Example 37.10 Find $\lim_{x\to 0^+}\left(\frac{1}{3x}-\frac{1}{x^2}\right)$.

This limit has form $\infty-\infty$. Get a common denominator and add the fractions:

$$\lim_{x\to 0^+}\left(\frac{1}{3x}-\frac{1}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{1}{3x}\frac{x}{x}-\frac{3}{3}\frac{1}{x^2}\right)=\lim_{x\to 0^+}\frac{x-3}{3x^2}=\boxed{-\infty.}$$

✍

37.3 The Indeterminate Forms 1^∞, ∞^0 and 0^0

This section investigates the final three intermediate forms: 1^∞, ∞^0 and 0^0. Limits having these forms have the structure $\lim_{x\to c} f(x)^{g(x)}$, that is, they are limits of a function raised to the power of another function.

Consider first 1^∞. We say a limit has this form if its structure is $\lim_{x\to c} f(x)^{g(x)}$, where $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = \infty$. For example, here are two very simple and transparent limits of form 1^∞. In each case the base function (2^x or 3^x) approaches 1 as $x \to 0$. But the exponent $1/x$ goes to ∞. Because $(a^x)^{1/x} = a$, these two limits are very easy to find.

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} (2^x)^{1/x} = 2 \\ \lim\limits_{x\to 0^+} (3^x)^{1/x} = 3 \end{array}\right\} \text{indeterminate form } 1^\infty$$

Next are two limits of indeterminate form ∞^0. Each is of form $\lim_{x\to c} f(x)^{g(x)}$, where $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = 0$. The expression ∞^0 alone is meaningless, but, as you can see below, *limits* of this form can be meanigful.

$$\left.\begin{array}{l} \lim\limits_{x\to \infty} (2^x)^{1/x} = 2 \\ \lim\limits_{x\to \infty} (3^x)^{1/x} = 3 \end{array}\right\} \text{indeterminate form } \infty^0$$

Finally here are two limits of form 0^0. By itself, 0^0 is undefined, but *limits* having form 0^0 can exist and can equal different numbers.

$$\left.\begin{array}{l} \lim\limits_{x\to 0} x^0 = 1 \\ \lim\limits_{x\to \infty} (2^{-x})^{1/x} = 2^{-1} = \dfrac{1}{2} \end{array}\right\} \text{indeterminate form } 0^0$$

Just one trick will handle limits of these three indeterminate forms. It relies on the natural logarithm function ln and its two properties $x = e^{\ln(x)}$ and $\ln(x^y) = y\ln(x)$.

The trick works like this. Suppose $\lim_{x\to c} f(x)^{g(x)}$ has form 1^∞, ∞^0 and 0^0. Using the above properties of ln this limit can be transformed as follows.

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln\left(f(x)^{g(x)}\right)} = \lim_{x\to c} e^{g(x)\cdot\ln(f(x))} = e^{\left(\lim_{x\to c} g(x)\cdot\ln(f(x))\right)}$$

Then you will find the the exponent $\lim_{x\to c} g(x)\cdot\ln\left(f(x)\right)$ has form $0\cdot\infty$ (or $\infty\cdot 0$), and this can be handled using the methods of the previous section.

Example 37.11 Find $\lim\limits_{x\to\infty} \sqrt[x]{x}$.

Write this as $\lim\limits_{x\to\infty} \sqrt[x]{x} = \lim\limits_{x\to\infty} x^{1/x}$, which has form ∞^0. Following our program,

$$\lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln(x^{1/x})} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(x)} = e^{\lim\limits_{x\to\infty} \frac{1}{x}\ln(x)}$$

The exponent $\lim\limits_{x\to\infty} \frac{1}{x}\ln(x)$ has form $0\cdot\infty$. By multiplying the $\frac{1}{x}$ in, we get

$$= e^{\lim\limits_{x\to\infty} \frac{\ln(x)}{x}}$$

Now the exponent has form $\frac{\infty}{\infty}$. Applying L'Hôpital's rule, this becomes

$$= e^{\lim\limits_{x\to\infty} \frac{1/x}{1}} = \lim_{x\to\infty} e^{1/x} = e^0 = \boxed{1.}$$ ✍

Example 37.12 Find $\lim\limits_{x\to 0} x^x$.

This has form 0^0. According to our plan for handling this, we write

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim\limits_{x\to 0^+} x\ln(x)}$$

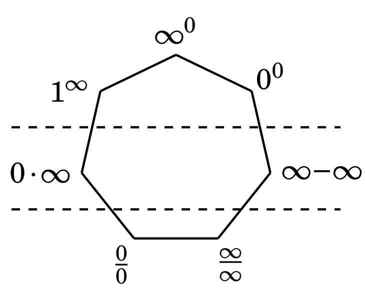
The exponent $\lim\limits_{x\to 0^+} x\ln(x)$ has form $0\cdot\infty$. Bringing the x down, we get

$$= e^{\lim\limits_{x\to 0^+} \frac{\ln(x)}{1/x}}$$

The exponent has form $\frac{\infty}{\infty}$. Applying L'Hôpital's rule, this becomes

$$= e^{\lim\limits_{x\to 0^+} \frac{1/x}{-1/x^2}} = \lim_{x\to 0^+} e^{-x} = e^0 = \boxed{1.}$$ ✍

In summary, we've seen seven indeterminate forms.



The forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ at the diagram's base are handled by L'Hôpital's rule. Those on the next tier, $0\cdot\infty$ and $\infty-\infty$ are handled by converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The three at the top are converted to $0\cdot\infty$ by exploiting the fact $x = e^{\ln(x)}$.

Exercises for Chapter 37

Evaluate the limits. Exercises 1–22 have form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

1. $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x}$

2. $\displaystyle\lim_{x\to 0}\frac{3x^2}{\cos(x)-1}$

3. $\displaystyle\lim_{x\to 0}\frac{8x^2}{\cos(x)-1}$

4. $\displaystyle\lim_{x\to 0}\frac{x^2}{\ln(\sec x)}$

5. $\displaystyle\lim_{x\to 1}\frac{e^x-e}{x^2-1}$

6. $\displaystyle\lim_{x\to \pi}\frac{\cos(x)+1}{(x-\pi)^2}$

7. $\displaystyle\lim_{x\to \infty}\frac{x^2+e^x}{x^2+x+e^x-1}$

8. $\displaystyle\lim_{x\to 1}\frac{\ln(x)}{x-1}$

9. $\displaystyle\lim_{x\to 0^+}\frac{\ln(x)}{e^{1/x}}$

10. $\displaystyle\lim_{x\to 0}\frac{4+2\ln|x|}{x-3\ln|x|}$

11. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{2x-\pi}{\cos(2\pi-x)}$

12. $\displaystyle\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$

13. $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\cos(x)}{\sin(2x)}$

14. $\displaystyle\lim_{x\to 0}\frac{3x}{1-e^{\sin x}}$

15. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

16. $\displaystyle\lim_{x\to \infty}\frac{\ln x}{\sqrt{x}}$

17. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$

18. $\displaystyle\lim_{x\to \pi}\frac{\cos(x)+1}{(x-\pi)^2}$

19. $\displaystyle\lim_{x\to \pi}\frac{\sin(x)}{x^2-\pi^2}$

20. $\displaystyle\lim_{x\to 0}\frac{e^{3x}-1}{x^2+3x}$

21. $\displaystyle\lim_{x\to 0}\frac{\ln|x|}{\csc(x)}$

22. $\displaystyle\lim_{x\to 1}\frac{\ln|x|}{4x-x^2-3}$

Exercises 23–36 have form $0\cdot\infty$ or $\infty-\infty$.

23. $\displaystyle\lim_{x\to 0} x\csc(x)$

24. $\displaystyle\lim_{x\to 0^+} x^2\ln x$

25. $\displaystyle\lim_{x\to \frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\sec(x)$

26. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

27. $\displaystyle\lim_{x\to \infty} x\sin\left(\frac{1}{5x}\right)$

28. $\displaystyle\lim_{x\to \infty}\left(xe^{-x}\right)$

29. $\displaystyle\lim_{x\to 0}\csc(7x)\sin(6x)$

30. $\displaystyle\lim_{x\to 0^+} x\ln(x)$

31. $\displaystyle\lim_{x\to \infty} x(e^{1/x}-1)$

32. $\displaystyle\lim_{x\to \infty}\big(\ln(2x)-\ln(x+1)\big)$

33. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2}\right)$

34. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)$

35. $\displaystyle\lim_{x\to 0^+}\big(\ln(\sin(x))-\ln(x)\big)$

36. $\displaystyle\lim_{x\to \infty}\left(\sqrt{x+1}-\sqrt{x}\right)$

Limits in Exercises 37–42 have one of the forms 0^0, 1^∞ or ∞^0.

37. $\displaystyle\lim_{x\to 0^+} x^x$

38. $\displaystyle\lim_{x\to 0}(1+x)^{\frac{1}{x}}$

39. $\displaystyle\lim_{x\to \infty}(\ln x)^{1/x}$

40. $\displaystyle\lim_{x\to \infty}(\ln(x))^{1/x}$

41. $\displaystyle\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x$

42. $\displaystyle\lim_{x\to 0^+}\left(e^x-1\right)^x$

For the remaining exercises, find the limits using any appropriate technique.

43. $\displaystyle\lim_{x\to\pi} \frac{4\pi^2 - 4x^2 + 1}{2\pi - 2x - 3}$

44. $\displaystyle\lim_{x\to\infty} \frac{2x - 5e^x}{3e^{2x} + 4x + 1}$

45. $\displaystyle\lim_{x\to\pi} \frac{4\pi^2 - 4x^2}{2\pi - 2x}$

46. $\displaystyle\lim_{x\to 0} \frac{\cos(2x) - \cos(x)}{\sin(x) + \cos(x) + x - 1}$

Exercise Solutions for Chapter 37

1. $\lim_{x\to 0}\frac{\sin(x^2)}{x}=\lim_{x\to 0}\frac{\cos(x^2)2x}{1}=\frac{\cos(0)2\cdot 0}{1}=0$

3. $\lim_{x\to 0}\frac{8x^2}{\cos(x)-1}=\lim_{x\to 0}\frac{16x}{-\sin(x)}=\lim_{x\to 0}\frac{16}{-\cos(x)}=\frac{16}{-\cos(0)}=-16$

5. $\lim_{x\to 1}\frac{e^x-e}{x^2-1}=\lim_{x\to 1}\frac{e^x}{2x}=\frac{e^1}{2\cdot 1}=\frac{e}{2}$

7. $\lim_{x\to\infty}\frac{x^2+e^x}{x^2+x+e^x-1}=\lim_{x\to\infty}\frac{2x+e^x}{2x+1+e^x}\lim_{x\to\infty}\frac{2+e^x}{2+e^x}=1$

9. $\lim_{x\to 0^+}\frac{\ln(x)}{e^{1/x}}=\lim_{x\to 0^+}\frac{1/x}{e^{1/x}(-x^{-2})}=\lim_{x\to 0^+}\frac{-x}{e^{1/x}}=0$ (Recall that $0/\infty=0$.)

11. $\lim_{x\to\frac{\pi}{2}}\frac{2x-\pi}{\cos(2\pi-x)}=\lim_{x\to\frac{\pi}{2}}\frac{2}{-\sin(2\pi-x)(-1)}=\lim_{x\to\frac{\pi}{2}}\frac{2}{\sin(2\pi-x)}=\frac{2}{\sin(3\pi/2)}=-2$

13. $\lim_{x\to\frac{\pi}{2}}\frac{\cos(x)}{\sin(2x)}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin(x)}{\cos(2x)2}=\frac{-\sin(\pi/2)}{2\cos(\pi)}=\frac{1}{2}$

15. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\frac{e^0}{2}=\frac{1}{2}$

17. $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$

19. $\lim_{x\to\pi}\frac{\sin(x)}{x^2-\pi^2}=\lim_{x\to\pi}\frac{\cos(x)}{2x}=\frac{\cos(\pi)}{2\pi}=-\frac{1}{2\pi}$

21. $\lim_{x\to 0}\frac{\ln|x|}{\csc(x)}=\lim_{x\to 0}\frac{1/x}{-\csc(x)\cot(x)}=\lim_{x\to 0}\frac{\sin(x)}{x}\frac{-1}{\cot(x)}=1\cdot 0=0$

23. $\lim_{x\to 0}x\csc(x)=\lim_{x\to 0}\frac{x}{\sin(x)}=\lim_{x\to 0}\frac{1}{\cos(x)}=\frac{1}{\cos(0)}=\frac{1}{1}=1$

25. $\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\sec(x)=\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)}{\cos(x)}=\lim_{x\to\frac{\pi}{2}}\frac{-1}{-\sin(x)}=\frac{1}{\sin(\pi/2)}=\frac{1}{1}=1$

27. $\lim_{x\to\infty}x\sin\left(\frac{1}{5x}\right)=\lim_{x\to\infty}\frac{\sin\left(\frac{1}{5x}\right)}{1/x}=\lim_{x\to\infty}\frac{\cos\left(\frac{1}{5x}\right)\frac{-1}{5x^2}}{-1/x^2}=\lim_{x\to\infty}\frac{\cos\left(\frac{1}{5x}\right)}{5}=\frac{\cos(0)}{5}=\frac{1}{5}$

29. $\lim_{x\to 0}\csc(7x)\sin(6x)=\lim_{x\to 0}\frac{\sin(6x)}{\sin(7x)}=\lim_{x\to 0}\frac{\cos(6x)6}{\cos(7x)7}=\frac{6\cos(0)}{7\cos(0)}=\frac{6}{7}$

31. $\lim_{x\to\infty}x(e^{1/x}-1)=\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}}=\lim_{x\to\infty}\frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}e^{1/x}=e^{\lim_{x\to\infty}\frac{1}{x}}=e^0=1$

33. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x-1}{x^2}=-\infty$

35. $\lim_{x\to 0^+}\big(\ln(\sin(x))-\ln(x)\big)=\lim_{x\to 0^+}\ln\left(\frac{\sin(x)}{x}\right)=\ln\left(\lim_{x\to 0^+}\frac{\sin(x)}{x}\right)=\ln(1)=0$

37. $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{\ln(x^x)} = \lim\limits_{x\to 0^+} e^{x\ln(x)} = \lim\limits_{x\to 0^+} e^{\frac{\ln(x)}{\frac{1}{x}}} = \lim\limits_{x\to 0^+} e^{\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = \lim\limits_{x\to 0^+} e^{-x} = e^0 = 1$

39. $\lim\limits_{x\to\infty} (\ln x)^{1/x} = \lim\limits_{x\to\infty} e^{\ln\left((\ln x)^{1/x}\right)} = \lim\limits_{x\to\infty} e^{\frac{1}{x}\ln(\ln x)} = \lim\limits_{x\to\infty} e^{\frac{\ln(\ln x)}{x}} = \lim\limits_{x\to\infty} e^{\frac{\frac{1/x}{\ln(x)}}{1}} = \lim\limits_{x\to\infty} e^{\frac{1}{x\ln(x)}} = e^0 = 1$

41. $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)} = \lim\limits_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = \lim\limits_{x\to\infty} e^{\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}} = \lim\limits_{x\to\infty} e^{\frac{\frac{-1/x^2}{1+\frac{1}{x}}}{-1/x^2}} = \lim\limits_{x\to\infty} e^{\frac{1}{1+\frac{1}{x}}} = e^{\frac{1}{1+0}} = e$

43. This is not indeterminate! Note that $\lim\limits_{x\to\pi} \dfrac{4\pi^2 - 4x^2 + 1}{2\pi - 2x - 3} = \dfrac{4\pi^2 - 4\pi^2 + 1}{2\pi - 2\pi - 3} = \dfrac{-1}{3}$

45. $\lim\limits_{x\to\pi} \dfrac{4\pi^2 - 4x^2}{2\pi - 2x} = \lim\limits_{x\to\pi} \dfrac{0 - 8x}{0 - 2} = 4\pi$