CHAPTER 28

Logarithmic Differentiation

We now have an impressive array of rules and techniques for finding derivatives of functions. But you don't have to look far for functions to which no rules apply. Consider the problem of differentiating

$$f(x) = x^x.$$

If this had the form $g(x) = x^a$ (that is, x to a constant power), then we'd use the power rule: $g'(x) = ax^{a-1}$. And if it were a constant to a variable power, $h(x) = a^x$, then we'd know that $h'(x) = \ln(a)a^x$. But x^x has neither of these forms. There's simply no rule for differentiating x^x.

This chapter develops a technique—called logarithmic differentiation—that transforms functions like x^x into forms to which existing rules apply.

The new technique is really nothing more than combining logarithm properties with implicit differentiation. So let's start with a review of relevant logarithm properties. Here are four that will be especially useful.

- $\ln\left(ab\right) = \ln(a) + \ln(b)$
- $\ln\left(a^b\right) = b\ln(a)$
- $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$
- $\ln\left(\dfrac{1}{b}\right) = -\ln(b)$

Actually, we'll need the following easily-checked and more robust versions of these rules, which can handle negative values of a and b.

- $\ln|ab| = \ln|a| + \ln|b|$
- $\ln|b^a| = a\ln|b|$
- $\ln\left|\dfrac{a}{b}\right| = \ln|a| - \ln|b|$
- $\ln\left|\dfrac{1}{b}\right| = -\ln|b|$

And we'll need our derivative rules for the natural logarithm.

- $\dfrac{d}{dx}\Big[\ln|x|\Big] = \dfrac{1}{x}$
- $\dfrac{d}{dx}\Big[\ln|g(x)|\Big] = \dfrac{1}{g(x)}g'(x)$

To motivate and explain logarithmic differentiation, let's return to our original dilemma:

Question: If $y = x^x$ what is $\frac{dy}{dx}$?

Since no rules apply, consider the following approach. Begin with the functional relationship

$$y = x^x$$

Take the natural log of both sides (first taking the absolute value to guard against potentially negative values, which are not in the domain of ln).

$$\ln|y| = \ln|x^x|$$

Next apply to this the property $\ln|a^b| = b\ln|a|$.

$$\ln|y| = x\ln|x|$$

The exponent has disappeared! Our normal derivative rules apply to either side of this equation. We can now differentiate each side. But remember that y is a function of x (in fact, $y = x^x$) so we have to use implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}\Big[\ln|y|\Big] &= \frac{d}{dx}\Big[x\ln|x|\Big] \\
\frac{1}{y}\frac{dy}{dx} &= 1\cdot\ln|x| + x\frac{1}{x} \\
\frac{1}{y}\frac{dy}{dx} &= \ln|x| + 1
\end{aligned}$$

The derivative $\frac{dy}{dx}$ that we seek has just come into the picture. We can solve for it by multiplying both sides of this equation by y.

$$\frac{dy}{dx} = y\left(\ln|x| + 1\right)$$

Finally, remember that $y = x^x$, so our derivative is

$$\frac{dy}{dx} = x^x\left(\ln|x| + 1\right).$$

Answer: The derivative of $y = x^x$ is $\frac{dy}{dx} = x^x\left(\ln|x| + 1\right)$.

Let's analyze what just happened. We had a function $y = f(x)$ that no derivative rules applied to. We took ln of both sides and simplified. Then we differentiated implicitly and solved for the derivative. This highly useful procedure is called **logarithmic differentiation.** Here is a summary.

> **Logarithmic Differentiation**
>
> To differentiate a complex or problematic function $f(x)$:
>
> 1. Write as $y = f(x)$
> 2. Take ln of both sides: $\ln|y| = \ln|f(x)|$
> 3. Simplify using log properties
> 4. Differentiate implicitly
> 5. Solve for $\frac{dy}{dx}$

Example 28.1 Differentiate the function $\sqrt{e^x(x+1)\cos(x)}$.

Actually we *could* do this the usual way, with the chain rule and the product rule. But the product rule would involve *three* functions, not just two. It would be a mess. Look at how logarithmic differentiation tames the process.

$$\begin{aligned}
y &= \sqrt{e^x(x+1)\cos(x)} \\
y &= \left(e^x(x+1)\cos(x)\right)^{1/2} && \text{(convert to a power)} \\
\ln|y| &= \ln\left|\left(e^x(x+1)\cos(x)\right)^{1/2}\right| && \text{(take ln of both sides)} \\
\ln|y| &= \frac{1}{2}\ln\left|e^x(x+1)\cos(x)\right| && \text{(simplify)} \\
\ln|y| &= \frac{1}{2}\left(\ln|e^x| + \ln|x+1| + \ln|\cos(x)|\right) \\
\ln|y| &= \frac{1}{2}\left(x + \ln|x+1| + \ln|\cos(x)|\right) \\
\frac{d}{dx}\Big[\ln|y|\Big] &= \frac{d}{dx}\left[\frac{1}{2}\left(x + \ln|x+1| + \ln|\cos(x)|\right)\right] && \text{(differentiate)} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\left(\frac{d}{dx}[x] + \frac{d}{dx}\Big[\ln|x+1|\Big] + \frac{d}{dx}\Big[\ln|\cos(x)|\Big]\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\left(1 + \frac{1}{x+1} - \frac{\sin(x)}{\cos(x)}\right) \\
\frac{dy}{dx} &= \frac{y}{2}\left(1 + \frac{1}{x+1} - \frac{\sin(x)}{\cos(x)}\right) && \text{(multiply by } y\text{)} \\
\frac{dy}{dx} &= \frac{\sqrt{e^x(x+1)\cos(x)}}{2}\left(1 + \frac{1}{x+1} - \frac{\sin(x)}{\cos(x)}\right) && ✍
\end{aligned}$$

We'll finish the chapter by tying up one lose end that has quietly dogged us since Chapter 17. There, when we proved the product rule $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$, we did so only for *positive integer values of* n. We said at the time that the product rule holds for all real values of n, and that we would eventually prove this fact. Ever since then we've been freely using the power rule with powers that are not positive integers.

It's time to prove that $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ holds for *any* real value of n. We can do this with logarithmic differentiation. Suppose that n is an arbitrary real number. We ask:

What is the derivative of $y = x^n$? (We want to show it really is nx^{n-1}.)

Let's apply logarithmic differentiation to this.

$$\begin{aligned}
y &= x^n & & \\
\ln|y| &= \ln|x^n| & & \text{(take ln of both sides)} \\
\ln|y| &= n\ln|x| & & \text{(simplify)} \\
\frac{d}{dx}\Big[\ln|y|\Big] &= \frac{d}{dx}\Big[n\ln|x|\Big] & & \text{(differentiate)} \\
\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{x} & & \\
\frac{dy}{dx} &= n\frac{y}{x} & & \text{(solve for } \tfrac{dy}{dx}\text{)} \\
\frac{dy}{dx} &= n\frac{x^n}{x^1} & & (y = x^n) \\
\frac{dy}{dx} &= nx^{n-1} & &
\end{aligned}$$

Therefore $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ holds for *any* real value of n. We now have full license to use the power rule.

Example 28.2 $\dfrac{d}{dx}\Big[x^{\pi}\Big] = \pi x^{\pi-1}$ ✍

Logarithmic differentiation is not a major technique, but it *is* occasionally useful. Working a few exercises may give you a greater appreciation of specialized tool.

Exercises for Chapter 28

Use logarithmic differentiation to find derivatives of the following functions.

1. $y=(5x+3)^x$

2. $f(x)=\sqrt{x}\sin^4(x)\ln(x)$

3. $y=x^{5x+3}$

4. $f(x)=\frac{e^x\sin(x)}{x^3\ln(x)}$

5. $f(x)=x^{\cos(x)}$

6. $f(x)=x^{\sin(x)}$

7. $f(x)=\big(\cos(x)\big)^x$

8. $f(x)=\big(\sin(x)\big)^x$

9. $y=(x^2+1)^x$

10. $y=(x^3+x)^x$

11. $y=x^{\ln(x)}$

12. $y=x^2\cos(x)\sin(x)$

13. $y=\sqrt{x}\sin(x)\cos(x)$

14. $y=\sqrt{e^x\tan(x)\big(x^2+x\big)}$

Exercise Solutions for Chapter 28

1. Differentiate $y=(5x+3)^x$.

$$\begin{aligned}
\ln|y| &= \ln\big|(5x+3)^x\big|\\
\ln|y| &= x\ln\big|5x+3\big|\\
D_x\Big[\ln|y|\Big] &= D_x\Big[x\ln\big|5x+3\big|\Big]\\
\frac{y'}{y} &= 1\cdot\ln\big|5x+3\big|+x\cdot\frac{5}{5x+3}\\
y' &= y\left(\ln\big|5x+3\big|+\frac{5x}{5x+3}\right)\\
y' &= (5x+3)^x\left(\ln\big|5x+3\big|+\frac{5x}{5x+3}\right)
\end{aligned}$$

3. Differentiate $y=x^{5x+3}$.

$$\begin{aligned}
\ln|y| &= \ln\big|x^{5x+3}\big|\\
\ln|y| &= (5x+3)\ln\big|x\big|\\
D_x\Big[\ln|y|\Big] &= D_x\Big[(5x+3)\ln\big|x\big|\Big]\\
\frac{y'}{y} &= 5\cdot\ln\big|x\big|+(5x+3)\cdot\frac{1}{x}\\
y' &= y\left(5\ln\big|x\big|+\frac{5x+3}{x}\right)\\
y' &= x^{5x+3}\left(5\ln\big|x\big|+\frac{5x+3}{x}\right)
\end{aligned}$$

5. Differentiate $f(x)=x^{\cos(x)}$.

$$\begin{aligned}
y &= x^{\cos(x)}\\
\ln\big|y\big| &= \ln\Big|x^{\cos(x)}\Big|\\
\ln\big|y\big| &= \cos(x)\ln|x|\\
\frac{d}{dx}\Big[\ln\big|y\big|\Big] &= \frac{d}{dx}\Big[\cos(x)\ln|x|\Big]\\
\frac{1}{y}\frac{dy}{dx} &= -\sin(x)\ln|x|+\cos(x)\frac{1}{x}\\
\frac{dy}{dx} &= y\left(-\sin(x)\ln|x|+\cos(x)\frac{1}{x}\right)\\
\frac{dy}{dx} &= x^{\cos(x)}\left(\sin(x)\ln\left|\frac{1}{x}\right|+\cos(x)\frac{1}{x}\right)
\end{aligned}$$

7. Differentiate $f(x)=\big(\cos(x)\big)^x$.

$$\begin{aligned}
y &= \big(\cos(x)\big)^x\\
\ln\big|y\big| &= \ln\big|\big(\cos(x)\big)^x\big|\\
\ln\big|y\big| &= x\ln\big|\cos(x)\big|\\
\frac{d}{dx}\Big[\ln\big|y\big|\Big] &= \frac{d}{dx}\Big[x\ln\big|\cos(x)\big|\Big]\\
\frac{1}{y}\frac{dy}{dx} &= 1\cdot\ln\big|\cos(x)\big|+x\frac{-\sin(x)}{\cos(x)}\\
\frac{dy}{dx} &= y\big(\ln\big|\cos(x)\big|-x\tan(x)\big)\\
\frac{dy}{dx} &= \big(\cos(x)\big)^x\big(\ln\big|\cos(x)\big|-x\tan(x)\big)
\end{aligned}$$

9. Differentiate $y = (x^2+1)^x$.

$$\begin{aligned}
y &= (x^2+1)^x \\
\ln|y| &= \ln|(x^2+1)^x| \\
\ln|y| &= x\ln|x^2+1| \\
\frac{d}{dx}\Big[\ln|y|\Big] &= \frac{d}{dx}\Big[x\ln|x^2+1|\Big] \\
\frac{1}{y}\frac{dy}{dx} &= 1\cdot\ln|x^2+1| + x\frac{2x}{x^2+1} \\
\frac{dy}{dx} &= y\left(\ln|x^2+1| + \frac{2x^2}{x^2+1}\right) \\
\frac{dy}{dx} &= (x^2+1)^x\left(\ln|x^2+1| + \frac{2x^2}{x^2+1}\right)
\end{aligned}$$

11. Differentiate $y = x^{\ln(x)}$.

Note: Since $\ln(x)$ appears here, we may assume x is in the domain of ln, so $x > 0$. Therefore $\ln|x| = \ln(x)$, and we will make this replacement below.

$$\begin{aligned}
y &= x^{\ln(x)} \\
\ln|y| &= \ln\left|x^{\ln(x)}\right| \\
\ln|y| &= \ln(x)\ln|x| \\
\ln|y| &= \ln(x)\ln(x) \\
\frac{d}{dx}\Big[\ln|y|\Big] &= \frac{d}{dx}\Big[\ln(x)\ln(x)\Big] \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}\ln(x) + \ln(x)\frac{1}{x} \\
\frac{dy}{dx} &= y2\ln(x)\frac{1}{x} \\
\frac{dy}{dx} &= \frac{2\ln(x)x^{\ln(x)}}{x}
\end{aligned}$$

13. Differentiate $y = \sqrt{x}\sin(x)\cos(x)$.

$$\begin{aligned}
\ln|y| &= \ln\left|\sqrt{x}\sin(x)\cos(x)\right| \\
\ln|y| &= \ln\left|x^{1/2}\sin(x)\cos(x)\right| \\
\ln|y| &= \ln\left|x^{1/2}\right| + \ln|\sin(x)| + \ln|\cos(x)| \\
\ln|y| &= \frac{1}{2}\ln|x| + \ln|\sin(x)| + \ln|\cos(x)| \\
D_x\Big[\ln|y|\Big] &= D_x\Big[\frac{1}{2}\ln|x| + \ln|\sin(x)| + \ln|\cos(x)|\Big] \\
\frac{y'}{y} &= \frac{1}{2}\cdot\frac{1}{x} + \frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}
\end{aligned}$$

$$
\begin{aligned}
y' &= y\left(\frac{1}{2x} + \cot(x) - \tan(x)\right) \\
y' &= \sqrt{x}\sin(x)\cos(x)\left(\frac{1}{2x} + \cot(x) - \tan(x)\right) \\
y' &= \frac{\sqrt{x}\sin(x)\cos(x)}{2x} + \cos^2(x) - \sin^2(x) \\
y' &= \frac{\sin(x)\cos(x)}{2\sqrt{x}} + \cos^2(x) - \sin^2(x)
\end{aligned}
$$