CHAPTER 24

Derivatives of Inverse Functions and Logarithms

We know that $\frac{d}{dx}\left[e^x\right] = e^x$. But what about derivatives of exponential function with bases other than e? In other words, what is $\frac{d}{dx}\left[a^x\right]$? And what about $\frac{d}{dx}\left[\ln(x)\right]$ and $\frac{d}{dx}\left[\log_a(x)\right]$? The main goal of this chapter is to answer these questions and thus expand our list of derivative rules.

Let's start with $\frac{d}{dx}\left[a^x\right]$. Since $\ln(x)$ is the inverse of e^x, we know $a = e^{\ln(a)}$. We can thus convert the power a^x to a power of e:

$$a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)x}.$$

With this, we can get the derivative of a^x with the chain rule:

$$\frac{d}{dx}\left[a^x\right] = \underbrace{\frac{d}{dx}\left[e^{\ln(a)x}\right] = e^{\ln(a)x}\frac{d}{dx}\left[\ln(a)x\right]}_{\text{chain rule}} = e^{\ln(a)x}\ln(a) = a^x\ln(a).$$

So the derivative of a^x is just a^x times the constant $\ln(a)$. This is a new rule.

$$\boxed{\textbf{Rule 16} \quad \frac{d}{dx}\left[a^x\right] = \ln(a)a^x}$$

For example, $\dfrac{d}{dx}\left[10^x\right] = \ln(10)10^x \approx 2.302 \cdot 10^x$. Also $\dfrac{d}{dx}\left[2^x\right] = \ln(2)2^x \approx 0.693 \cdot 2^x$. Notice how special the base e is: $\dfrac{d}{dx}\left[e^x\right] = \ln(e)e^x = 1 \cdot e^x = e^x$. The base $a = e$ is the only base for which the derivative of a^x is 1 times a^x.

Next we will get a rule for $\frac{d}{dx}\left[\ln(x)\right]$. Our strategy will be to use the fact that $\ln(x)$ is the inverse of e^x, that is,

$$\text{if } f(x) = e^x, \text{ then } f^{-1}(x) = \ln(x).$$

Our plan is to first develop a general rule for $\frac{d}{dx}\left[f^{-1}(x)\right]$ and then use it to get $\frac{d}{dx}\left[\ln(x)\right]$. (See Chapter 4 If you need to review inverses.)

Thus our immediate question is: What is $\frac{d}{dx}\left[f^{-1}(x)\right]$?

To answer this, think about the relation between f and its inverse f^{-1}:

$$f\left(f^{-1}(x)\right) = x.$$

The two sides of this equation are equal functions, so if we differentiate both sides the derivatives will be equal:

$$\frac{d}{dx}\left[f\left(f^{-1}(x)\right)\right] = \frac{d}{dx}\left[x\right]$$

The right-hand side of this equation is 1. The left-hand side is the derivative of a composition, so we can apply the chain rule to it:

$$f'\left(f^{-1}(x)\right)\frac{d}{dx}\left[f^{-1}(x)\right] = 1$$

In applying the chain rule we multiplied $f'\left(f^{-1}(x)\right)$ by $\frac{d}{dx}\left[f^{-1}(x)\right]$. We stopped there because we don't know what $\frac{d}{dx}\left[f^{-1}(x)\right]$ is. But it's exactly what we want to find! We can isolate it by dividing the above equation by $f'\left(f^{-1}(x)\right)$:

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right).}$$

This is our latest rule.

Rule 17 (The inverse rule) If f is a function having a derivative f' and an inverse f^{-1}, then

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

To illustrate this rule, suppose $f(x) = x^3$, which has an inverse $f^{-1}(x) = \sqrt[3]{x}$. Let's find $\frac{d}{dx}\left[f^{-1}(x)\right]$. We know that $f'(x) = 3x^2$, so our new rule gives

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{3\left(f^{-1}(x)\right)^2} = \boxed{\frac{1}{3\sqrt[3]{x^2}}}.$$

Granted, this is not all that impressive, since we can use the power rule to get the same answer:

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{d}{dx}\left[\sqrt[3]{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{3}}\right] = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \boxed{\frac{1}{3\sqrt[3]{x^2}}}.$$

But the inverse rule can be very useful. We'll now use it to find the derivative of $\ln(x)$. Say $f(x) = e^x$, so $f^{-1}(x) = \ln(x)$. Then

$$\begin{aligned}
\frac{d}{dx}\Big[\ln(x)\Big] &= \frac{d}{dx}\Big[f^{-1}(x)\Big] && \text{because } \ln(x) = f^{-1}(x) \\
&= \frac{1}{f'\big(f^{-1}(x)\big)} && \text{by inverse rule} \\
&= \frac{1}{f'\big(\ln(x)\big)} && \text{because } f^{-1}(x) = \ln(x) \\
&= \frac{1}{e^{\ln(x)}} && \text{becase } f'(x) = e^x \\
&= \frac{1}{x} && \text{because } e^{\ln(x)} = x.
\end{aligned}$$

Thus $\dfrac{d}{dx}\Big[\ln(x)\Big] = \dfrac{1}{x}$. Figure 24.1 (left) illustrates this remarkable fact. It shows the function $f(x) = \ln(x)$ along with its derivative $f'(x) = \frac{1}{x}$. Notice how if x is near 0, the tangent to $\ln(x)$ at x is very steep, and indeed the derivative $\frac{1}{x}$ is very large. But as x gets bigger, the tangent to $\ln(x)$ gets closer to horizontal (slope 0) while the derivative $\frac{1}{x}$ approaches zero.

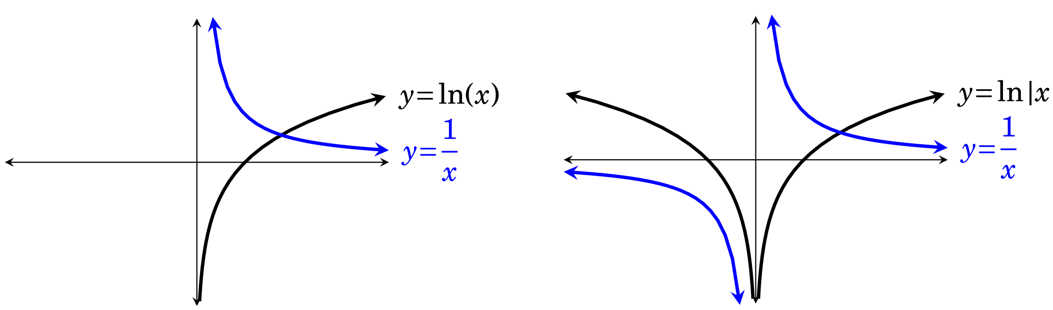

Figure 24.1. Left: the graphs of $f(x) = \ln(x)$ (black) and $f'(x) = \frac{1}{x}$ (blue) with domain $(0,\infty)$. Right: the graphs of $f(x) = \ln|x|$ (black) and $f'(x) = \frac{1}{x}$ (blue).

Notice however, that the domain of $\ln(x)$ is $(0,\infty)$ but the domain of $\frac{1}{x}$ is $(-\infty,0)\cup(0,\infty)$. So when we say that the derivative of $\ln(x)$ is $\frac{1}{x}$, we really mean $\frac{1}{x}$ with its domain restricted to $(0,\infty)$. Figure 24.1 (right) shows a somewhat more complete scenario. It shows the function $\ln(|x|)$, which we will abbreviate as $\ln|x|$. This function has domain $(-\infty,0)\cup(0,\infty)$, and its derivative is $\frac{1}{x}$ with its usual domain. So our latest rule has two parts.

Rule 18 $\quad \dfrac{d}{dx}\Big[\ln(x)\Big] = \dfrac{1}{x} \quad$ and $\quad \dfrac{d}{dx}\Big[\ln|x|\Big] = \dfrac{1}{x}.$

Here it is understood that in the first formula the domain of $\ln(x)$ and $\frac{1}{x}$ is $(0,\infty)$. In the second formula the domain of $\ln|x|$ and $\frac{1}{x}$ is all real numbers except 0. Do not sweat the difference between the two versions of this rule – they say almost the same thing, and the second implies the first. We will mostly use the first version in parts 3 and 4 of this book, but the second version becomes particularly useful in Part 5.

At the beginning of this chapter we said our main goals were to find formulas for $\frac{d}{dx}[a^x]$, $\frac{d}{dx}[\ln(x)]$ and $\frac{d}{dx}[\log_a(x)]$. We've done all but the last one. For it we will use the change of base formula (Fact 5.1 in Chapter 5, page 88) which states

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Using this, the constant multiple rule and Rule 18, we get

$$\frac{d}{dx}\Big[\log_a(x)\Big] = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(a)}\right] = \frac{1}{\ln(a)}\cdot\frac{d}{dx}\Big[\ln(x)\Big] = \frac{1}{\ln(a)}\cdot\frac{1}{x}.$$

With our prior agreement about domains, we get another two-part formula.

Rule 19 $\frac{d}{dx}\Big[\log_a(x)\Big] = \frac{1}{x\ln(a)}$ and $\frac{d}{dx}\Big[\log_a|x|\Big] = \frac{1}{x\ln(a)}$.

Example 24.1 $\frac{d}{dx}\Big[\log_3(x)\tan(x)\Big] = \frac{d}{dx}\Big[\log_3(x)\Big]\tan(x) + \log_3(x)\frac{d}{dx}\Big[\tan(x)\Big]$

(product rule) $= \boxed{\frac{1}{x\ln(3)}\tan(x) + \log_3(x)\sec^2(x)}$. ✍

Example 24.2 Find $\frac{d}{dx}\Big[\sqrt{5+x^3+\ln(x)}\,\Big]$.

This is the derivative of a function to a power, so we can use the generalized power rule:

$$\begin{aligned}
\frac{d}{dx}\Big[\sqrt{5+x^3+\ln(x)}\,\Big] &= \frac{d}{dx}\Big[\big(5+x^3+\ln(x)\big)^{1/2}\Big] \\
&= \frac{1}{2}\big(5+x^3+\ln(x)\big)^{1/2-1}\frac{d}{dx}\Big[5+x^3+\ln(x)\Big] \\
&= \frac{1}{2}\big(5+x^3+\ln(x)\big)^{-1/2}\left(3x^2+\frac{1}{x}\right) \\
&= \boxed{\frac{3x^2+\frac{1}{x}}{2\sqrt{5+x^3+\ln(x)}}}.
\end{aligned}$$

✍

.

Example 24.3
$$\frac{d}{dx}\left[7+x+\big(\ln(x)\big)^3\right] = 0+1+3\big(\ln(x)\big)^2\frac{d}{dx}\big[\ln(x)\big]$$
$$= 1+3\big(\ln(x)\big)^2\frac{1}{x} = \boxed{1+\frac{3\big(\ln(x)\big)^2}{x}}. \quad ✍$$

Example 24.4
$$\frac{d}{dx}\left[\frac{\ln(x)}{x}\right] = \frac{\frac{d}{dx}\big[\ln(x)\big]\cdot x+\ln(x)\cdot\frac{d}{dx}\big[x\big]}{x^2} = \frac{\frac{1}{x}\cdot x+\ln(x)\cdot 1}{x^2}$$

(quotient rule)
$$= \boxed{\frac{1+\ln(x)}{x^2}}. \quad ✍$$

Example 24.5 Find the derivative of 10^{x^2+3x+2}.

The composition $y=10^{x^2+3x+2}$ can be broken up as $\begin{cases} y=10^u \\ u=x^2+3x+2. \end{cases}$

The chain rule then gives
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \ln(10)\,10^u\cdot(2x+3+0)\\ &= \boxed{\ln(10)\,10^{x^2+3x+2}(2x+3)}.\end{aligned} \quad ✍$$

In Example 24.5 we differentiated a function of form $a^{g(x)}$. Let's repeat our steps to get a chain rule generalization for the rule $\frac{d}{dx}\big[a^x\big]=\ln(a)a^x$.

Example 24.6 Find the derivative of $a^{g(x)}$.

The composition $y=a^{g(x)}$ can be broken up as $\begin{cases} y=a^u \\ u=g(x). \end{cases}$

The chain rule then gives
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \ln(a)a^u\cdot g'(x)\\ &= \boxed{\ln(a)a^{g(x)}g'(x)}.\end{aligned} \quad ✍$$

This examples shws $\frac{d}{dx}\big[a^{g(x)}\big]=\ln(a)a^{g(x)}g'(x)$, a companion to the rule $\frac{d}{dx}\big[e^{g(x)}\big]=e^{g(x)}g'(x)$. We will summarize these and chain rule generalizations of the other rules from this chapter on the bottom of the next page.

Example 24.7 Find the derivative of $y = \ln\left|\sin(x)\right|$.

This is a composition, and the function can be broken up as $\begin{cases} y = \ln|u| \\ u = \sin(x) \end{cases}$

The chain rule gives $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{u}\cos(x) = \dfrac{1}{\sin(x)}\cos(x) = \boxed{\dfrac{\cos(x)}{\sin(x)}}$. ✍

Example 24.7 illustrates a common pattern, which is to differentiate a function of from $\ln\left|g(x)\right|$ or $\ln\left(g(x)\right)$. Let's redo the example in this setting.

Example 24.8 Find the derivative of $y = \ln\left|g(x)\right|$.

This is a composition, and the function can be broken up as $\begin{cases} y = \ln|u| \\ u = g(x) \end{cases}$

The chain rule gives $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{u}g'(x) = \dfrac{1}{g(x)}g'(x) = \boxed{\dfrac{g'(x)}{g(x)}}$. ✍

Example 24.8 has shown that

$$\frac{d}{dx}\Big[\ln\left|g(x)\right|\Big] = \frac{1}{g(x)} \cdot \frac{d}{dx}\Big[g(x)\Big] = \frac{g'(x)}{g(x)}.$$

This is the chain rule generalization of the rule $\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}$, and it is worth remembering. It implies $\frac{d}{dx}\left[\ln\left(g(x)\right)\right] = \frac{g'(x)}{g(x)}$, and we often us it this way. (Recall $\ln\left(g(x)\right)$ is not defined when $g(x)$ is negative, so the rule as stated for $\ln\left|g(x)\right|$ is more all-encompassing.)

Here is a summary of this chapter's main rules, along side their chain rule generalizations. Remember them and internalize them.

Differentiation rules for exponential and log functions

Rule	Chain rule generalization
$\dfrac{d}{dx}\Big[e^x\Big] = e^x$	$\dfrac{d}{dx}\Big[e^{g(x)}\Big] = e^{g(x)}g'(x)$
$\dfrac{d}{dx}\Big[a^x\Big] = \ln(a)a^x$	$\dfrac{d}{dx}\Big[a^{g(x)}\Big] = \ln(a)a^{g(x)}g'(x)$
$\dfrac{d}{dx}\Big[\ln\left|x\right|\Big] = \dfrac{1}{x}$	$\dfrac{d}{dx}\Big[\ln\left|g(x)\right|\Big] = \dfrac{g'(x)}{g(x)}$
$\dfrac{d}{dx}\Big[\log_a\left|x\right|\Big] = \dfrac{1}{x\ln(a)}$	$\dfrac{d}{dx}\Big[\log_a\left|g(x)\right|\Big] = \dfrac{g'(x)}{g(x)\ln(a)}$

We prefer the base e, so you should expect to that the formulas for a^x and $\log_a$ to play less of a role. (Though in computer science, $\log_2$ is significant!)

Example 24.9 Find the derivative of $y = \ln\left|4x^5 + 6x^3 + x + 3\right|$.

We will do this in two different ways. First we will use the chain rule, and then we will use the formula $\frac{d}{dx}\left[\ln|g(x)|\right] = \frac{g'(x)}{g(x)}$ from the previous page.

Using the chain rule, we first break the function up as $\begin{cases} y = \ln|u| \\ u = 4x^5 + 6x^3 + x + 3 \end{cases}$

The chain rule gives
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u}\left(20x^4 + 18x^2 + 1\right) \\ &= \frac{1}{4x^5+6x^3+x+3}\left(20x^4+18x^2+1\right) = \boxed{\frac{20x^4 + 18x^2 + 1}{4x^5 + 6x^3 + x + 3}}.\end{aligned}$$

Next, using the formula $\frac{d}{dx}\left[\ln|g(x)|\right] = \frac{g'(x)}{g(x)}$, from the previous page, the answer comes in one step: $\frac{d}{dx}\left[\ln\left|4x^5 + 6x^3 + x + 3\right|\right] = \boxed{\frac{20x^4 + 18x^2 + 1}{4x^5 + 6x^3 + x + 3}}$.

So using the formula is quicker. But you should soon reach the point where the above to approaches are equally automatic. Doing the chain rule in your head is in essence using the formula. ✍

Example 24.10 Find the derivative of $y = \ln\left|\tan\left(\sqrt{x^2 + 3x}\right)\right|$.

This has the form of a composition $\ln|g(x)|$, so we can use either the straight chain rule or the formula $\frac{d}{dx}\left[\ln|g(x)|\right] = \frac{1}{g(x)}g'(x)$ from the previous page. Let's try the formula.

$$\begin{aligned}\frac{d}{dx}\left[\ln\left|\tan\left(\sqrt{x^2+3x}\right)\right|\right] &= \frac{1}{\tan\left(\sqrt{x^2+3x}\right)}\frac{d}{dx}\left[\tan\left(\sqrt{x^2+3x}\right)\right] \\ &= \frac{1}{\tan\left(\sqrt{x^2+3x}\right)}\sec^2\left(\sqrt{x^2+3x}\right)\frac{d}{dx}\left[\sqrt{x^2+3x}\right] \\ &= \frac{1}{\tan\left(\sqrt{x^2+3x}\right)}\sec^2\left(\sqrt{x^2+3x}\right)\frac{1}{2}\left(x^2+3x\right)^{\frac{1}{2}-1}\frac{d}{dx}\left[x^2+3x\right] \\ &= \frac{1}{\tan\left(\sqrt{x^2+3x}\right)}\sec^2\left(\sqrt{x^2+3x}\right)\frac{1}{2}\left(x^2+3x\right)^{-\frac{1}{2}}(2x+3) \\ &= \boxed{\frac{(2x+3)\sec^2\left(\sqrt{x^2+3x}\right)}{2\tan\left(\sqrt{x^2+3x}\right)\sqrt{x^2+3x}}}\end{aligned}$$

✍

Exercises for Chapter 24

In exercises 1–20 differentiate the given function.

1. $\ln(x)+\frac{1}{x}+\sqrt{x}+3$

2. $\ln\left(x^2+\frac{1}{x}\right)$

3. $\frac{\ln(w)}{w}$

4. $\frac{1}{x^2+\ln(x)}$

5. $\ln\left(\sin^3(x)\right)$

6. $\ln\left(\tan(x)\right)$

7. $5+\ln(\pi\theta)+\sqrt{\theta^3}$

8. $\ln\left(\sec\left(x^3\right)\right)$

9. $\cos\left(\ln(x)\right)$

10. $\left(\sec\left(\ln x\right)\right)^3$

11. $\pi^2+\ln(5\theta)+\sqrt{\theta}^9$

12. $\ln\left(\sec(x^3)\right)$

13. $\ln(x^2+1)\sqrt{3x+1}$

14. $\sec\left(\ln(x^3)\right)$

15. $\ln\left(xe^x\right)$

16. $\ln(x)e^x$

17. $\tan\left(\ln(x)\right)+x$

18. $\ln\left(\sin\left(x^3\right)\right)$

19. $\ln\left(1+\frac{1}{x}\right)$

20. $\frac{x^3\ln(x)}{x^3+1}$

21. Find $\lim\limits_{h\to 0}\frac{\ln(3+h)-\ln(3)}{h}$

22. Find $\lim\limits_{z\to 3}\frac{2^z-8}{z-3}$

Exercise Solutions for Chapter 24

In exercises 1–20 differentiate the given function.

1. $\frac{d}{dx}\left[\ln(x)+\frac{1}{x}+\sqrt{x}+3\right]=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{2\sqrt{x}}$

3. $\frac{d}{dw}\left[\frac{\ln(w)}{w}\right]=\frac{\frac{1}{w}\cdot w-\ln(w)\cdot 1}{w^2}=\frac{1-\ln(w)}{w^2}$

5. $\frac{d}{dx}\left[\ln\left(\sin^3(x)\right)\right]=\frac{1}{\sin^3(x)}\frac{d}{dx}\left[\sin^3(x)\right]=\frac{1}{\sin^3(x)}3\sin^2(x)\frac{d}{dx}\left[\sin(x)\right]=3\frac{\cos(x)}{\sin(x)}$

7. $\frac{d}{d\theta}\left[5+\ln(\pi\theta)+\sqrt{\theta^3}\right]=\frac{d}{d\theta}\left[5+\ln(\pi\theta)+\theta^{3/2}\right]=0+\frac{\pi}{\pi\theta}+\frac{3}{2}\theta^{1/2}=\frac{1}{\theta}+\frac{3}{2}\sqrt{\theta}$

9. $\frac{d}{dx}\left[\cos\left(\ln(x)\right)\right]=-\sin\left(\ln(x)\right)\frac{1}{x}=-\frac{\sin\left(\ln(x)\right)}{x}$

11. $\frac{d}{dx}\left[\pi^2+\ln(5\theta)+\sqrt{\theta}^9\right]=\frac{d}{dx}\left[\pi^2+\ln(5\theta)+\theta^{9/2}\right]=0+\frac{5}{5\theta}+\frac{9}{2}\theta^{7/2}=\frac{1}{\theta}+\frac{9}{2}\sqrt{\theta}^7$

13. $$\frac{d}{dx}\left[\ln(x^2+1)\sqrt{3x+1}\right] = \frac{d}{dx}\left[\ln(x^2+1)\right]\sqrt{3x+1} + \ln(x^2+1)\frac{d}{dx}\left[\sqrt{3x+1}\right]$$
$$= \frac{2x}{x^2+1}\sqrt{3x+1} + \ln(x^2+1)\frac{3}{2\sqrt{3x+1}}$$

15. $$\frac{d}{dx}\left[\ln\left(xe^x\right)\right] = \frac{1}{xe^x}\frac{d}{dx}\left[xe^x\right] = \frac{1}{xe^x}\left(1\cdot e^x + xe^x\right) = \frac{e^x(1+x)}{xe^x} = \frac{1+x}{x}$$

17. $$\frac{d}{dx}\left[\tan\left(\ln(x)\right)+x\right] = \sec^2\left(\ln(x)\right)\frac{1}{x}+1 = \frac{\sec^2\left(\ln(x)\right)}{x}+1$$

19. $$\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right] = \frac{1}{1+\frac{1}{x}}\frac{d}{dx}\left[1+\frac{1}{x}\right] = \frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right) = \frac{-1}{x^2+x}$$

21. Find $\lim\limits_{h\to 0}\dfrac{\ln(3+h)-\ln(3)}{h}$. Solution: Let $f(x)=\ln(x)$.

Then $\lim\limits_{h\to 0}\dfrac{\ln(x+h)-\ln(x)}{h} = f'(x) = \dfrac{1}{x}$. Thus $\lim\limits_{h\to 0}\dfrac{\ln(3+h)-\ln(3)}{h} = f'(3) = \boxed{\dfrac{1}{3}.}$