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MATH 200 MIDTERM EXAM — OCT. 27, 2021

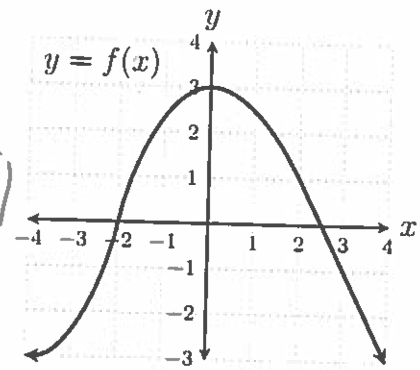
Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (10 points) Answer the questions about the function f graphed below.

(a) $\lim_{z\to 3} \frac{f(z)-f(3)}{z-3} = f'(3) = \boxed{-2}$ (slope of tangent at $(3, f(3))$)

(b) $\lim_{x\to 0} \frac{1}{3-f(x)} = \boxed{\infty}$

approaching 0 positive



(c) $\lim_{x\to\infty} f\left(2+\frac{1}{x}\right) = f\left(\lim_{x\to\infty}\left(2+\frac{1}{x}\right)\right) = f(2+0)$
$= f(2) = \boxed{1}$

(d) $\lim_{x\to -2} \frac{\sin(f(x))}{f(x)} = \boxed{1}$

(e) $\lim_{x\to -2} \frac{\sin(f(x))}{f(x)+1} = \frac{\lim_{x\to -2} \sin(f(x))}{\lim_{x\to -2}(f(x)+1)} = \frac{\sin(0)}{0+1} = \frac{0}{1} = \boxed{0}$

2. (20 points) Find the limits

(a) $\lim_{x\to 0} \tan^{-1}(x-1) = \tan^{-1}\left(\lim_{x\to 0}(x-1)\right) = \tan^{-1}(0-1) = \tan^{-1}(-1) = \boxed{\frac{-\pi}{4}}$

(b) $\lim_{x\to \pi/2} e^{\cos(x)} = e^{\lim_{x\to\pi/2}\cos(x)} = e^{\cos(\pi/2)} = e^{0} = \boxed{1}$

(c) $\lim_{x\to 3} \frac{x^2-7x+12}{3x-9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-4)}{3\cancel{(x-3)}} = \lim_{x\to 3} \frac{x-4}{3} = \frac{3-4}{3} = \boxed{-\frac{1}{3}}$

$\frac{0}{0}$ so try to cancel

(d) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-2} = \frac{\lim_{x\to 4}(\sqrt{x}-2)}{\lim_{x\to 4}(x-2)} = \frac{\sqrt{4}-2}{4-2} = \frac{2-2}{2} = \frac{0}{2} = \boxed{0}$

Not $\frac{0}{0}$!!!

3. (7 points) Use a **limit definition** of the derivative to find the derivative of $f(x) = \dfrac{1}{1-x}$.

$$f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x} = \lim_{z \to x} \frac{\frac{1}{1-z} - \frac{1}{1-x}}{z-x}$$

$$= \lim_{z \to x} \frac{\frac{1}{1-z} - \frac{1}{1-x}}{z-x} \cdot \frac{(1-z)(1-x)}{(1-z)(1-x)}$$

$$= \lim_{z \to x} \frac{(1-x)-(1-z)}{(z-x)(1-z)(1-x)}$$

$$= \lim_{z \to x} \frac{z-x}{(z-x)(1-z)(1-x)} = \lim_{z \to x} \frac{1}{(1-z)(1-x)}$$

$$= \frac{1}{(1-x)(1-x)} = \boxed{\frac{1}{(1-x)^2}}$$

4. (7 points) Suppose $f(x) = x^3 - 3x$ and $g(x) = 3x^2 + 6x$. Find all x for which the tangent to $y = f(x)$ at $(x, f(x))$ is parallel to the tangent to $y = g(x)$ at $(x, g(x))$.

Parallel tangents have equal slopes, so we need to solve

$$f'(x) = g'(x)$$
$$3x^2 - 3 = 6x + 6$$
$$3x^2 - 6x - 9 = 0$$
$$3(x^2 - 2x - 3) = 0$$
$$3(x+1)(x-3) = 0$$
$$x = -1 \qquad x = 3$$

Answer: Tangents are parallel at $x = -1$ and $x = 3$

5. (7 points) An object moving on a straight line is $s(t) = t^3 - 3t^2$ feet from its starting point at time t seconds. Find its acceleration when its velocity is -3 feet per second.

Velocity at time t is $v(t) = s'(t) = 3t^2 - 6t$ ft/sec

To find when velocity is -3 ft/sec we solve the equation

$$v(t) = -3$$
$$3t^2 - 6t = -3$$
$$3t^2 - 6t + 3 = 0$$
$$3(t^2 - 2t + 1) = 0$$
$$3(t-1)(t-1) = 0$$

Solution is $t = 1$, so velocity is -3 ft/sec at time $t = 1$ second

Acceleration at time t is $a(t) = v'(t) = 6t - 6$. So at time $t = 1$ acceleration is $a(1) = 6 \cdot 1 - 6 = \boxed{0 \text{ ft/sec}^2}$

6. (35 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

(a) $f(x) = \sqrt{2}x^2 + e$ $\quad f'(x) = \sqrt{2}\cdot 2x + 0 = \boxed{2\sqrt{2}x}$

(b) $f(x) = x\ln|x| - x$ $\quad f'(x) = 1\cdot \ln|x| + x\frac{1}{x} - 1$

$= \ln|x| + 1 - 1 = \boxed{\ln|x|}$

(c) $f(x) = e^{\sec(x)}$ $\quad f'(x) = \boxed{e^{\sec(x)}\sec(x)\tan(x)}$

(d) $f(x) = e^x\sec(x)$ $\quad f'(x) = \boxed{e^x\sec(x) + e^x\sec(x)\tan(x)}$

(e) $f(x) = \left(\frac{x+1}{x-1}\right)^3$ $\quad f'(x) = 3\left(\frac{x+1}{x-1}\right)^{3-1}\frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2}$

$= 3\left(\frac{x+1}{x-1}\right)^2\frac{-2}{(x-1)^2} = \boxed{-6\frac{(x+1)^2}{(x-1)^4}}$

(f) $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$ $\quad f'(x) = -\frac{1}{2}(1-x)^{-3/2}(-1)$

$= \frac{1}{2(1-x)^{3/2}} = \boxed{\frac{1}{2\sqrt{1-x}^3}}$

(g) $y = \cos^2(\ln(x^3+x)) = (\cos(\ln|x^3+x|))^2$

$y' = 2\cos(\ln(x^3+x))^1 D_x[\cos(\ln(x^3+x))]$

$= \boxed{2\cos(\ln(x^3+x))(-\sin(\ln(x^3+x)))\frac{3x^2+1}{x^3+x}}$

7. (7 points) Given the equation $xy^3 = xy + 6$, find y'.

$$D_x\left[xy^3\right] = D_x\left[xy + 6\right]$$

$$1 \cdot y^3 + x\, 3y^2 y' = 1 \cdot y + xy' + 0$$

$$3xy^2y' - xy' = y - y^3$$

$$y'(3xy^2 - x) = y - y^3$$

$$\boxed{y' = \frac{y - y^3}{3xy^2 - x}}$$

8. (7 points) Find the derivative of $f(x) = x^x$. (use logarithmic differentiation)

$$y = x^x$$

$$\ln|y| = \ln|x^x|$$

$$\ln|y| = x \ln|x|$$

$$D_x\left[\ln|y|\right] = D_x\left[x \ln|x|\right]$$

$$\frac{y'}{y} = 1 \cdot \ln|x| + x\frac{1}{x}$$

$$y' = y(\ln|x| + 1)$$

$$\boxed{y' = x^x(\ln|x| + 1)}$$