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# MATH 200 – FINAL EXAM

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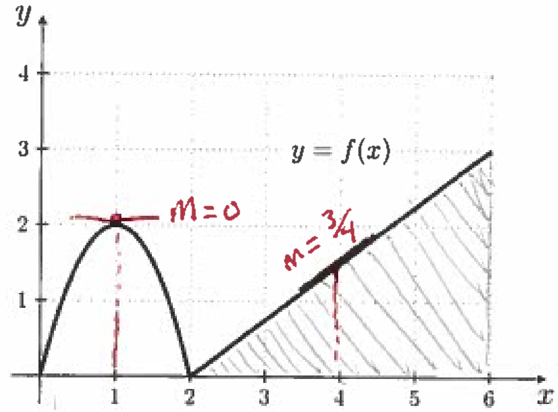
**Directions.** Answer the questions in the space provided. This is a closed-notes, closed book exam. No calculators, no computers and no formula sheets.
For numeric answers, give exact, simplified quantities. ($\sqrt{2}$ instead of 2.14, etc.).
Put your final answer in a $\boxed{\text{box}}$ when appropriate. You have three hours.

(1) (10 points) Answer the following questions involving the function $f(x)$ graphed below.




(a) $f'(1) = \boxed{0}$

(b) $f'(4) = \boxed{\frac{3}{4}}$

(c) $\displaystyle\lim_{x\to 1} \frac{3x+1-2f(x)}{x^2-3x-2} = \frac{3\cdot 1+1-2f(1)}{1^2-3\cdot 1-2} = \frac{3+1-2\cdot 2}{-4} = \frac{0}{-4} = \boxed{0}$

(d) $\displaystyle\lim_{x\to 4} \frac{2f(x)-3}{x^2-2x-8} = \lim_{x\to 4} \frac{2f'(x)-0}{2x-2-0} = \frac{2f'(4)}{2\cdot 4-2} = \frac{2\cdot\frac{3}{4}}{6} = \boxed{\frac{1}{4}}$

form $\frac{0}{0}$

apply L'Hôpital

(e) $\displaystyle\int_2^6 f(x)\,dx =$ (shaded area above) $= \frac{1}{2}4\cdot 3 = \boxed{6}$

(2) (12 points) Find the limits. Please show work.

(a) $\lim\limits_{x\to\pi} \dfrac{5}{2+\sin(x)} = \dfrac{5}{2+\sin(\pi)} = \dfrac{5}{2+0} = \boxed{\dfrac{5}{2}}$

(b) $\lim\limits_{x\to 5} \dfrac{\sqrt{x+4}-3}{x-5} = \lim\limits_{x\to 5} \dfrac{\sqrt{x+4}-3}{x-5} \cdot \dfrac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$

$= \lim\limits_{x\to 5} \dfrac{\sqrt{x+4}^2 - 9}{(x-5)(\sqrt{x+4}+3)} = \lim\limits_{x\to 5} \dfrac{\cancel{x-5}}{(\cancel{x-5})(\sqrt{x+4}+3)}$

$= \lim\limits_{x\to 5} \dfrac{1}{\sqrt{x+4}+3} = \dfrac{1}{\sqrt{5+4}+3} = \dfrac{1}{\sqrt{9}+3} = \dfrac{1}{3+3} = \boxed{\dfrac{1}{6}}$

(c) $\lim\limits_{x\to 0} \dfrac{e^{2x}-2x-1}{x^2} = \lim\limits_{x\to 0} \dfrac{e^{2x}\cdot 2 - 2}{2x} = \lim\limits_{x\to 0} \dfrac{e^{2x}\cdot 4}{2} = \dfrac{e^0 \cdot 4}{2} = \boxed{2}$

form $\frac{0}{0}$

form $\frac{0}{0}$

apply L'Hôpital

apply L'Hopital

(d) $\lim\limits_{x\to\infty} x\tan\left(\frac{3}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\tan\left(\frac{3}{x}\right)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{\sec^2\left(\frac{3}{x}\right)\left(\frac{-3}{x^2}\right)}{\frac{-1}{x^2}}$

form $\infty \cdot 0$

form $\frac{0}{0}$

$= \lim\limits_{x\to\infty} \sec^2(3/x)\, 3$

$= \lim\limits_{x\to\infty} \dfrac{3}{\cos^2(3/x)}$

$= \dfrac{3}{\cos^2(0)} = \dfrac{3}{1} = \boxed{3}$

(3) (18 points) Find the derivatives of the following functions.

(a) $f(x) = 5x^6 + \frac{3}{x} - 7\sin(x) + 2$

$\boxed{f'(x) = 30x^5 - \frac{3}{x^2} - 7\cos(x)}$

(b) $f(x) = x^3 \cos(x)$

$\boxed{f'(x) = 3x^2 \cos(x) - x^3 \sin(x)}$

(c) $p(z) = \frac{8z^5}{e^z}$

$p'(z) = \frac{40z^4 e^z - 8z^5 e^z}{(e^z)^2} = \frac{e^z(40z^4 - 8z^5)}{e^z \cdot e^z}$

$= \boxed{\frac{40z^4 - 8z^5}{e^z}}$

(d) $f(x) = \ln(20x^3 - 7x)$

$\boxed{f'(x) = \frac{60x^2 - 7}{20x^3 - 7x}}$

(e) $h(x) = \tan^{-1}(5x^2)$

$h'(x) = \frac{1}{1 + (5x^2)^2} \; 10x$

$= \boxed{\frac{10x}{1 + 25x^4}}$

(f) $f(x) = (1 + \tan^4(x))^3$

$f'(x) = 3(1 + \tan^4(x))^2 \, 4\tan^3(x) \sec^2(x)$

$= \boxed{12(1 + \tan^4(x))^2 \tan^3(x) \sec^2(x)}$

(4) (5 points) Given the equation $y^2 + 9xy = 2x^4$, find $y'$.

$\{ y = f(x) \}$

$$\frac{d}{dx}\left[y^2 + 9xy\right] = \frac{d}{dx}\left[2x^4\right]$$

$$2yy' + 9y + 9xy' = 8x^3$$

$$2yy' + 9xy' = 8x^3 - 9y$$

$$y'(2y + 9x) = 8x^3 - 9y$$

$$\boxed{y' = \frac{8x^3 - 9y}{2y + 9x}}$$

(5) (5 points) Find the value of $c$ for which the following function is continuous at $\frac{\pi}{4}$.

$$f(x) = \begin{cases} \sin^2(x) + c & \text{if } x < \frac{\pi}{4} \\ 1 + \frac{cx}{\pi} & \text{if } x \geq \frac{\pi}{4} \end{cases}$$

$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} \sin^2(x) + c = \sin^2\left(\frac{\pi}{4}\right) + c$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 + c = \frac{1}{2} + c$$

$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \pi/4^+} \left(1 + \frac{cx}{\pi}\right) = 1 + \frac{c\pi/4}{\pi} = 1 + \frac{c}{4}$$

For continuity at $x = \frac{\pi}{4}$, these two limits must be equal, so $\frac{1}{2} + c = 1 + \frac{c}{4}$

$$\frac{3}{4}c = \frac{1}{2}$$

$$\boxed{c = 2/3}$$

(6) (10 points) Determine whether the following statements are **true** <u>or</u> **false**. Explain.

(a) If $f'(c) = 0$ then $f$ must have a local extremum at $c$.

$\boxed{\text{FALSE}}$ Think of $f(x) = x^3$ and $c = 0$. Then $f'(0) = 3 \cdot 0^2 = 0$, but there is no local extremum at $x=0$.

(b) If $f'(x) < 0$ and $f''(x) > 0$ on an interval, then $f$ is decreasing at an increasing rate.

$\boxed{\text{TRUE}}$ $f'(x)<0$ means $f$ is decreasing. Also $f'(x)$ is the rate of change of $f(x)$, and $f''(x)>0$ means the rate $f'(x)$ <u>increases</u>.

(c) $\int (x^2-1)^2 \, dx = \frac{(x^2-1)^3}{3} + C$.

Let's check: $\frac{d}{dx}\left[\frac{(x^2-1)^3}{3} + c\right] = \frac{1}{3} 3(x^2-1)^2 2x = (x^2-1)^2 2x$

$\neq (x^2-1)^2$. Therefore this is $\boxed{\text{FALSE}}$

(d) If $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to a^-} f(x) = f(a)$, then $f$ is continuous at $a$.

This means $\lim_{x\to a^+} f(x) = f(a) = \lim_{x\to a^-} f(x)$, so $\lim_{x\to a} f(x) = f(a)$, meaning $f$ is continuous at $a$. $\boxed{\text{TRUE}}$

(e) If the acceleration of an object is increasing, then its velocity is also increasing.

$\boxed{\text{FALSE}}$ Imagine $a(t) = 2t-2$, which is <u>increasing</u>. Then $v(t) = t^2 - 2t$, which <u>decreases</u> on $(0, 1)$

(7) (5 points) Find $\frac{d}{dx}\left[\int_0^{x^2} \frac{1}{(t+2)^3}\, dt\right]$.

The function $\int_0^{x^2} \frac{1}{(t+2)^3} dt$ is a composition:

$$\begin{cases} y = \int_0^u \frac{1}{(t+2)^3} dt \\ u = x^2 \end{cases}$$

Then by the chain rule

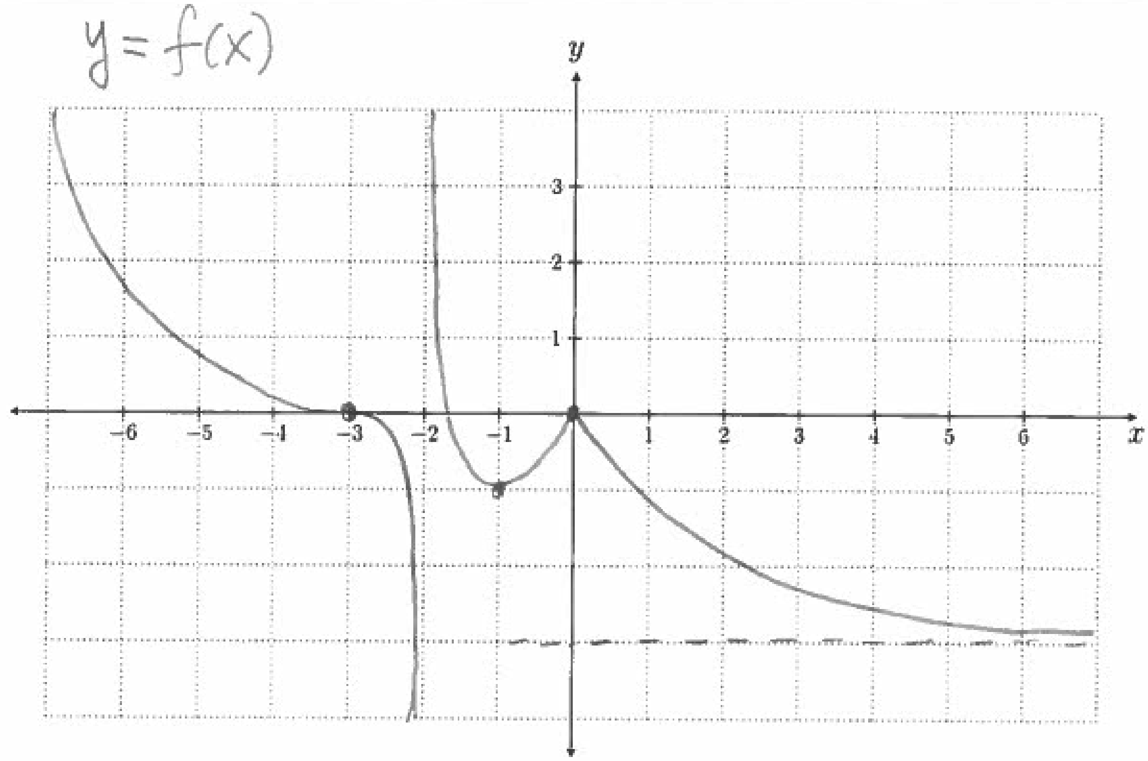
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{(u+2)^3} 2x = \boxed{\frac{2x}{(x^2+2)^3}}$$

(8) (8 points) Draw a graph of $y = f(x)$ meeting **all** of the following conditions.

- $f$ is continuous on $(-\infty, -2) \cup (-2, \infty)$
- $f$ is differentiable on $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$.
- $\lim_{x \to -2^-} f(x) = -\infty$ and $\lim_{x \to -2^+} f(x) = +\infty$
- $\lim_{x \to -\infty} f(x) = +\infty$ and $\lim_{x \to \infty} f(x) = -3$
- $f$, $f'$ and $f''$ meet the the conditions in the following table:

| $x$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|
| $f(x)$ | 0 | DNE | $-1$ | 0 |
| $f'(x)$ | 0 | DNE | 0 | DNE |
| $f''(x)$ | 0 | DNE | | DNE |

- $f'(x) < 0$ on $(-\infty, -2) \cup (-2, -1) \cup (0, \infty)$,
- $f'(x) > 0$ on $(-1, 0)$,
- $f''(x) < 0$ on $(-3, -2)$,
- $f''(x) > 0$ on $(-\infty, -3) \cup (-2, 0) \cup (0, \infty)$.



(9) (4 points) Suppose the derivative of a function $f(x)$ is $f'(x) = (x+3)^2(x-2)(x+1)^2$

(a) Find the intervals where $f(x)$ is increasing/decreasing.

Since $(x+3)^2 > 0$ and $(x+1)^2 > 0$, the factor $(x-2)$ controls the sign of $f'(x)$.

$$\overset{2}{- - - - \mid + + + + + +} \quad f'(x) = (x+3)^2(x-2)(x+1)^2$$

$\boxed{\begin{array}{l} f \text{ increases on } (2, \infty) \\ f \text{ decreases on } (-\infty, 2) \end{array}}$

(b) List any local extrema of $f(x)$. Specify whether it is a maximum or minimum.

$\boxed{f(x) \text{ has a local minimum at } x = 2}$ and $\boxed{\text{no local maximum}}$ by 1st derivative test.

(10) (8 points) This problem concerns three functions $f$, $g$ and $h$.

At $x = 2$, the graph of $y = f(x)$ has tangent line $y = 3x + 4$.

At $x = -1$, the graph of $y = g(x)$ has tangent line $y = -x + 1$.

Suppose $h(x) = f(g(x))$.

Answer the following questions using the above information.

(a) $f(2) = 3 \cdot 2 + 4 = \boxed{10}$

(b) $f'(2) = \boxed{3}$

(c) $g(-1) = -(-1) + 1 = \boxed{2}$

(d) $g'(-1) = \boxed{-1}$

(e) $h(-1) = f(g(-1)) = f(2) = \boxed{10}$

(f) $h'(-1) = f'(g(-1))\, g'(-1) = f'(2) \cdot (-1) = 3(-1) = \boxed{-3}$

(g) Find the tangent line to the graph of $y = h(x)$ at $x = -1$.

$$y - y_0 = m(x - x_0)$$

$$y - h(-1) = h'(-1)(x - (-1))$$

$$y - 10 = -3(x+1) \longrightarrow \boxed{y = -3x + 7}$$

(11) (9 points) Find the following indefinite integrals.

(a) $\displaystyle\int \left(2x^3 + \frac{5}{x} + \frac{1}{x^5} - \pi\right) dx$

$$= 2\frac{x^4}{4} + 5\ln|x| + \frac{1}{-5+1}x^{-5+1} - \pi x + C$$

$$= \boxed{\frac{x^4}{2} + 5\ln|x| - \frac{1}{4x^4} - \pi x + C}$$

(b) $\displaystyle\int \frac{x^3}{\sqrt{x^4+5}}\, dx$ $\displaystyle = \int (x^4+5)^{-1/2} x^3\,dx = \int u^{-1/2}\frac{1}{4}\,du$

$u = x^4 + 5$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$$= \frac{1}{4}\int u^{-1/2}\,du$$

$$= \frac{1}{4}\frac{1}{\frac{-1}{2}+1}u^{-\frac{1}{2}+1} + C$$

$$= \frac{1}{4}\frac{1}{1/2}u^{1/2} + C = \frac{1}{2}\sqrt{u} + C$$

$$= \boxed{\frac{1}{2}\sqrt{x^4+5} + C}$$

(c) $\displaystyle\int \sin^2(\theta)\cos(\theta)\, d\theta$

$$= \int (\sin(\theta))^2\cos(\theta)\,d\theta = \int u^2\,du$$

$u = \sin(\theta)$

$\frac{du}{d\theta} = \cos(\theta)$

$du = \cos(\theta)\,d\theta$

$$= \frac{u^3}{3} + C$$

$$= \frac{(\sin(\theta))^3}{3} + C$$

$$= \boxed{\frac{\sin^3(\theta)}{3} + C}$$

(12) (6 points) Compute the following definite integrals.

(a) $\int_{-1}^{1} (6x^5 - 12x^3)\, dx$

$$= \left[ 6\frac{x^6}{6} - 12\frac{x^4}{4} \right]_{-1}^{1}$$

$$= \left[ x^6 - 3x^4 \right]_{-1}^{1}$$

$$= \left(1^6 - 3(1)^4\right) - \left((-1)^6 - 3(-1)^4\right)$$

$$= (1-3) - (1-3) = \boxed{0}$$

(b) $\int_{0}^{1} 3x^2(x^3-1)^4\, dx$

$$= \int_0^1 (x^3-1)^4\, 3x^2\, dx$$

$u = x^3 - 1$

$\frac{du}{dx} = 3x^2$

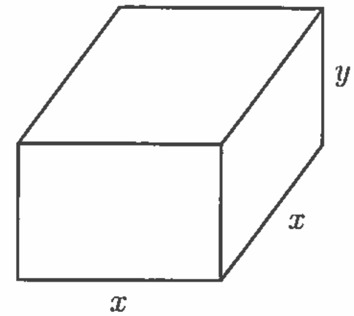
$du = 3x^2\, dx$

$$= \int_{0^3-1}^{1^3-1} u^4\, du$$

$$= \int_{-1}^{0} u^4\, du = \left[ \frac{u^5}{5} \right]_{-1}^{0}$$

$$= \frac{0^4}{5} - \frac{(-1)^5}{5} = \boxed{\frac{1}{5}}$$

(13) (10 points) A tank with a square base is to be constructed to hold 10,000 cubic feet of water. The metal top costs \$6 per square foot, and the concrete sides and bottom cost \$4 per square foot. What dimensions $x$ and $y$ yield the lowest cost of materials?




$$\begin{aligned} \text{Cost} &= \text{bottom} + \text{top} + \text{sides} \\ &= 4x^2 + 6x^2 + 4 \cdot 4xy \\ &= 10x^2 + 16xy \\ &= 10x^2 + 16x\frac{10000}{x^2} \\ &= 10x^2 + \frac{160000}{x} \end{aligned}$$

Constraint:

Volume $= x^2 y$

$10000 = x^2 y$

$y = \dfrac{10000}{x^2}$

$$\boxed{\text{Cost} = C(x) = 10x^2 + \frac{160000}{x} \quad \text{Minimize this on } (0,\infty)}$$

$$C'(x) = 20x - \frac{160000}{x^2} = 0$$

$$20x = \frac{160000}{x^2}$$

$$20x^3 = 160000$$

$$x^3 = 8000$$

$$x = \sqrt[3]{8000} = 20$$

Critical Point


20

$- - - - \mid + + + + \quad C'(x) = 20x - \dfrac{160000}{x^2}$

Find $y$:

$x = 20$

$y = \dfrac{10000}{20^2} = 25$

$$\boxed{\text{Answer: Dimensions } x = 20,\ y = 25 \text{ will minimize cost}}$$