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MATH 200 – FINAL EXAM
R. Hammack December 17, 2021

This is a closed-notes, closed book exam. No calculators, no computers, etc. Put phones away. Answer the questions in the space provided. Put your final answer in a $\boxed{\text{box}}$ when appropriate. Each numbered problem is 4 points.

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1. Find $\displaystyle\lim_{x\to 0}\frac{x+\sin(x)}{x}$ $= \displaystyle\lim_{x\to 0}\left(\frac{x}{x}+\frac{\sin(x)}{x}\right) = \lim_{x\to 0}\left(1+\frac{\sin(x)}{x}\right) = 1+1 = \boxed{2}$

Alternatively, use L'Hopital: $\displaystyle\lim_{x\to 0}\frac{x+\sin(x)}{x} = \lim_{x\to 0}\frac{1+\cos(x)}{1}$

form $\frac{0}{0}$

$= \dfrac{1+\cos(0)}{1} = \dfrac{1+1}{1} = \boxed{2}$

2. Find $\displaystyle\lim_{x\to 2^+}\frac{|2-x|}{2-x}$ $= \displaystyle\lim_{x\to 2^+}\frac{-(2-x)}{2-x} = \lim_{x\to 2^+}(-1) = \boxed{-1}$

$2 \leftarrow x$

Note: $x>2$ so $2-x$ is negative, hence $|2-x| = -(2-x)$

3. Find $\displaystyle\lim_{x\to\infty}\frac{2e^x+3}{x+e^x}$ $= \displaystyle\lim_{x\to\infty}\frac{2e^x}{1+e^x} = \lim_{x\to\infty}\frac{2\cancel{e^x}}{\cancel{e^x}} = \boxed{2}$

form $\frac{\infty}{\infty}$ form $\frac{\infty}{\infty}$ (Using L'Hopital twice)

4. Find $\displaystyle\lim_{x\to\pi^-}(x-\pi)\tan(x/2)$ $= \displaystyle\lim_{x\to\pi^-}\frac{x-\pi}{\frac{1}{\tan(x/2)}}$

form $0\cdot\infty$

$= \displaystyle\lim_{x\to\pi^-}\frac{x-\pi}{\cot(x/2)} = \lim_{x\to\pi^-}\frac{-1}{-\csc^2(x/2)}$

form $\frac{0}{0}$

$= \sin^2(\pi/2) = \boxed{1}$

5. Find the derivative: $f(x) = x^2 + \frac{1}{x^2} + \pi^2 \quad = x^2 + x^{-2} + \pi^2$

$$f'(x) = 2x - 2x^{-2-1} + 0 = \boxed{2x - \frac{2}{x^3}}$$

6. Find the derivative: $y = \tan\left(x^2 - 2x - 7\right)$

$$y' = \boxed{\sec^2(x^2-2x-7)\,(2x-2)}$$

7. Find the derivative: $f(x) = \left(\frac{\cos(x)}{1+x^5}\right)^5$

$$f'(x) = 5\left(\frac{\cos(x)}{1+x^5}\right)^4 D_x\left[\frac{\cos(x)}{1+x^5}\right]$$

$$= \boxed{5\left(\frac{\cos(x)}{1+x^5}\right) \frac{-\sin(x)(1+x^5) - \cos(x)\cdot 5x^4}{(1+x^5)^2}}$$

8. Find the derivative: $y = x^2 \ln\left(3 + x + x^2\right)$

$$y' = 2x \cdot \ln(3+x+x^2) + x^2 \frac{0+1+2x}{3+x+x^2}$$

$$= \boxed{2x \ln(3+x+x^2) + \frac{x^2+2x^3}{3+x+x^2}}$$

9. Find the derivative: $y = \int_3^{x^2+1} \sqrt{1+\sqrt{t}}\, dt$

$$\begin{cases} y = \int_3^u \sqrt{1+\sqrt{t}}\, dt \\ u = x^2+1 \end{cases}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sqrt{1+\sqrt{u}}\,(2x+0) = \boxed{\sqrt{1+\sqrt{x^2+1}}\cdot 2x}$$

$$= \boxed{2x\sqrt{1+\sqrt{x^2+1}}}$$

10. Find the indefinite integral: $\int \left(3x^2 + \frac{1}{x^2} + 3\right)\, dx$ $= \int (3x^2 + x^{-2} + 3)\, dx$

$= 3\frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} + 3x + C = \boxed{x^3 - \frac{1}{x} + 3x + C}$

11. Find the indefinite integral: $\int (x^5 + 5x + 11)^{11} (x^4 + 1)\, dx$

$u = x^5 + 5x + 11$

$\frac{du}{dx} = 5x^4 + 5$

$du = (5x^4 + 5)dx$

$\frac{1}{5}du = (x^4 + 1)dx$

$= \int u^{11} \frac{1}{5}\, du = \frac{1}{5}\int u^{11}\, du$

$= \frac{1}{5} \frac{u^{11+1}}{11+1} + C = \frac{u^{12}}{60} + C$

$= \boxed{\frac{(x^5 + 5x + 11)^{12}}{60} + C}$

12. Find the definite integral: $\int_1^2 (x^2 + 2x + 1)\, dx$

$= \left[\frac{x^3}{3} + x^2 + x\right]_1^2 = \left(\frac{2^3}{3} + 2^2 + 2\right) - \left(\frac{1^3}{3} + 1^2 + 1\right)$

$= \left(\frac{8}{3} + 6\right) - \left(\frac{1}{3} + 2\right) = \frac{7}{3} + 4 = \frac{7}{3} + \frac{12}{3} = \boxed{\frac{19}{3}}$

13. Find the definite integral: $\int_{-1}^0 \frac{x}{1 + x^2}\, dx$ $= \int_{-1}^{0} \frac{1}{1+x^2}\, x\, dx$

$u = 1 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{1}{2}du = x\, dx$

$= \int_{1+(-1)^2}^{1+0^2} \frac{1}{u}\, \frac{1}{2}\, du$

$= \frac{1}{2}\int_2^1 \frac{1}{u}\, du = \frac{1}{2}\Big[\ln|u|\Big]_2^1$

$= \frac{1}{2}\ln|1| - \frac{1}{2}\ln|2| = \boxed{-\frac{1}{2}\ln|2|}$

14. Use the limit definition of a derivative to compute the derivative of $f(x) = \dfrac{1}{x}$.

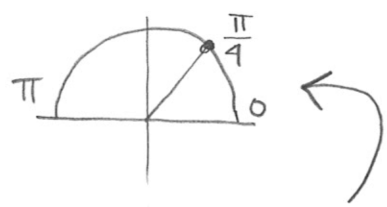
$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\frac{1}{z} - \frac{1}{x}}{z - x}$$

$$= \lim_{z \to x} \frac{\frac{1}{z} - \frac{1}{x}}{z - x} \cdot \frac{zx}{zx} = \lim_{z \to x} \frac{x - z}{(z - x)zx}$$

$$= \lim_{z \to x} \frac{-(z - x)}{(z - x)zx} = \lim_{z \to x} \frac{-1}{zx}$$

$$= \frac{-1}{x \cdot x} = -\frac{1}{x^2} \qquad \boxed{\text{Thus } f'(x) = \frac{-1}{x^2}}$$

15. Find the global maximum and minimum values of the function $f(x) = \sin(x) + \cos(x)$ on the interval $[0, \pi]$.



$f'(x) = \cos(x) - \sin(x) = 0$

The only value of x in $[0, \pi]$ for which $f'(x) = 0$ is $x = \pi/4$ (see unit circle)

Thus the only critical point in $[0, \pi]$ is $x = \pi/4$

$f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$

$f(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ ← MAX

$f(\pi) = \sin(\pi) + \cos(\pi) = 0 - 1 = -1$ ← MIN

$$\boxed{\begin{array}{ll} \text{The global maximum is } \sqrt{2} & \text{at } x = \pi/4 \\ \text{The global minimum is } -1 & \text{at } x = \pi \end{array}}$$

The problems on this page concern the function $f(x) = x + \dfrac{1}{x}$. ← (Domain $(-\infty,0)\cup(0,\infty)$)

16. Find the critical points of $f(x)$.

$$f'(x) = 1 - \frac{1}{x^2} = 0$$
$$1 = \frac{1}{x^2}$$
$$x^2 = 1$$
$$x = \pm 1$$

$\boxed{\text{The critical points are } x = 1 \text{ and } x = -1}$

(Note $x = 0$ is not a critical point because 0 is not in the domain of $f(x)$)

17. Find the intervals on which $f(x)$ is increasing/decreasing.

$-1 \qquad 0 \qquad 1$

$+ + + + \mid - - - - \; - - - - \mid + + +$ $\quad f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2} = \dfrac{(x-1)(x+1)}{x^2}$

f increases on $(-\infty, -1) \cup (1, \infty)$
f decreases on $(-1, 0) \cup (0, 1)$

18. Identify the locations of any local extrema of $f(x)$

By 1st derivative test (and # 17 above)

f has a local maximum at $x = -1$
f has a local minimum at $x = 1$

19. Find the intervals on which $f(x)$ is concave up/down.

$$f'(x) = 1 - \frac{1}{x^2} \rightsquigarrow f''(x) = \frac{2}{x^3}$$

0

$- - - - - - - - \mid + + + + + +$ $f'(x)$

f is concave down on $(-\infty, 0)$
f is concave up on $(0, \infty)$

20. Given the equation $x^2 + xy^2 = y$, use implicit differentiation to find y'.

$$D_x\left[x^2+xy^2\right] = D_x[y]$$

$$2x + 1\cdot y^2 + x\cdot 2yy' = y'$$

$$2xyy' - y' = -y^2 - 2x$$

$$y'(2xy - 1) = -y^2 - 2x$$

$$y' = \boxed{\frac{-y^2-2x}{2xy-1}} = \boxed{\frac{y^2+2x}{1-2xy}}$$

21. Find the area of the region contained under the graph of $f(x) = \sqrt{x} + 1$, between $x = 1$ and $x = 4$.

$$\int_1^4 (\sqrt{x}+1)\,dx = \int_1^4 \left(x^{1/2}+1\right)dx$$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x\right]_1^4 = \left[\frac{x^{3/2}}{3/2} + x\right]_1^4$$

$$= \left[\frac{2x^{3/2}}{3} + x\right]_1^4 = \left[\frac{2\sqrt{x}^3}{3} + x\right]_1^4$$

$$= \left(\frac{2\sqrt{4}^3}{3} + 4\right) - \left(\frac{2\sqrt{1}^3}{3} + 1\right) = \left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 1\right)$$

$$= \frac{14}{3} + 3 = \frac{14}{3} + \frac{9}{3} = \boxed{\frac{23}{3} \text{ square units}}$$

22. A ball, tossed straight up, has a constant acceleration of -32 feet per second per second. At time $t = 0$ its velocity is $v(0) = 10$ feet per second, and its position is $s(0) = 6$ feet. Find the position function $s(t)$.

$$v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C$$

Then $10 = v(0) = -32 \cdot 0 + C$, so $C = 10$

and $\boxed{v(t) = -32t + 10}$

$$s(t) = \int v(t)\,dt = \int (-32t + 10)\,dt$$

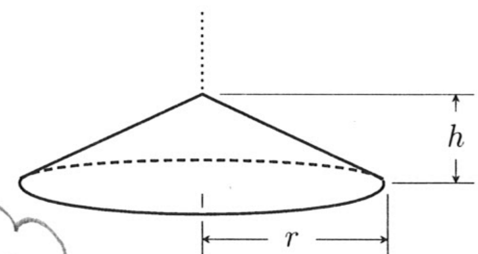
$$= -32\frac{t^2}{2} + 10t + C = -16t^2 + 10t + C.$$

Thus $s(t) = -16t^2 + 10t + C$

Know $6 = s(0) = -16 \cdot 0^2 + 10 \cdot 0 + C$, so $C = 6$

Consequently $\boxed{s(t) = -16t^2 + 10t + 6}$

23. Sand falls at a rate of 4 cubic feet per minute, making a conical pile whose height h is always half its radius r. Find the rate of change of the radius r (in feet/min) when $r = 2$ feet.



Know $\frac{dV}{dt} = 4$

Want $\frac{dr}{dt}$ (when $r = 2$)

$$V = \frac{1}{3}\pi r^2 h \quad \leftarrow h = \frac{1}{2}r$$

$$V = \frac{1}{3}\pi r^2 \frac{1}{2} r$$

$$V = \frac{\pi}{6} r^3$$

$$D_t[V] = D_t\left[\frac{\pi}{6} r^3\right]$$

$$\frac{dV}{dt} = \frac{\pi}{6} 3r^2 \frac{dr}{dt}$$

$$\rightarrow 4 = \frac{\pi}{2} r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{4}{\frac{\pi}{2} r^2} = \frac{8}{\pi r^2}$$

Ans $\left.\frac{dr}{dt}\right|_{r=2} = \frac{8}{\pi \cdot 2^2} = \boxed{\frac{2}{\pi}\ \text{ft/min}}$

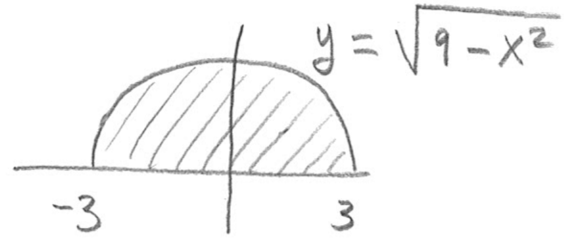
Geometry formula: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

24. Find $\int_0^3 \sqrt{9-x^2}\,dx$. (Hint: consider this as area under a graph.)

$$\int_0^3 \sqrt{9-x^2}\,dx = \left(\begin{matrix}\text{Shaded}\\ \text{Area}\end{matrix}\right)$$

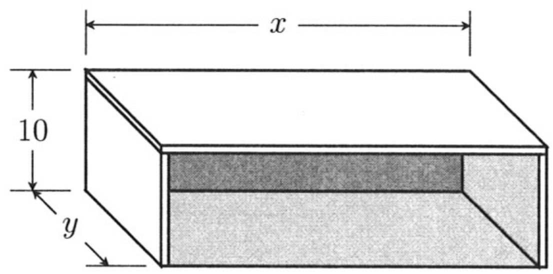
$$= \frac{1}{2}\pi 3^2$$

$$= \boxed{\frac{9\pi}{2}}$$



(half of a circle of radius 3)

25. A 10-foot-high shed with flat roof and three walls is to be constructed, as illustrated below. It is required that the shed have 800 square feet of floor space. Materials for the roof cost \$2 per square foot. Materials for the walls cost \$1 per square foot. Materials for the floor cost \$3 per square foot. What dimensions x and y will minimize the total cost of materials?



Cost = (walls) + (roof) + (floor)

Cost = \$1(10y + 10y + 10x) + \$2xy + \$3xy

Cost = 20y + 10x + 5xy

Constraint:

floor = 800 sq ft = xy

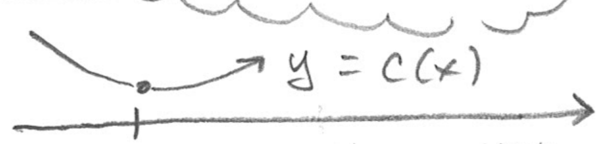
$800 = xy$

$y = \frac{800}{x}$

$$\text{Cost} = 20\cdot\frac{800}{x} + 10x + 5x\frac{800}{x}$$

$$C(x) = \frac{16000}{x} + 10x + 4000$$

Find global minimum of $C(x)$ on $(0, \infty)$



$- - - \underset{40}{|} + + +$ $C'(x) = -\frac{16000}{x^2} + 10$

$$C'(x) = -\frac{16000}{x^2} + 10 = 0$$

$$\frac{-16000}{x^2} = -10$$

$$x^2 = 1600$$

$$x = \sqrt{1600} = 40 \leftarrow \text{critical point!}$$

Answer Cost minimized when $x = 40$ $y = \frac{800}{40} = 20$