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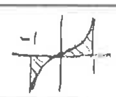
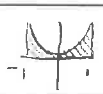
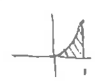
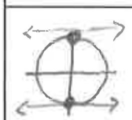
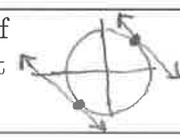
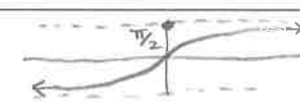
MATH 200 – Final Exam
VCU Sections 1,2,4,5
December 12, 2012

Directions. Answer the questions in the space provided. This is a closed-notes, closed book exam; no calculators, no computers and no formula sheets. You have three hours.

1. (Warmup) Fill in each entry in the middle column with one of the symbols $<$, $>$, or $=$ to express the relationship between the quantities to either side. Rows may be solved quickly using an understanding of course topics, or more slowly by brute force computation.

Left column	$>$ $=$ $<$	Right column
$\frac{\sqrt{2}}{2} = \sin \pi/4$	$<$	$\tan \pi/4 = 1$
$3.14 \approx \pi$	$>$	$e \approx 2.718$
 $\int_{-1}^{1} x^3\,dx$	$<$	$\int_{0}^{1} x^3\,dx$ 
 $\int_{-1}^{1} x^2\,dx$	$>$	$\int_{0}^{1} x^2\,dx$ 
$f'(x) = e x^{e-1}$ $f'(1) = e \cdot 1^{e-1} = e$ $f'(1)$ where $f(x) = x^e$	$=$	$g'(1)$ where $g(x) = e^x$ $g'(x) = e^x$ $g'(1) = e^1 = e$
$f'(1000) = e(1000)^{e-1}$ $< e \cdot 1000^2$ $< 3 \cdot 1000000$ $f'(1,000)$ where $f(x) = x^e$	$<$	$g'(1,000)$ where $g(x) = e^x$ $g'(x) = e^x$ $g'(1000) = e^{1000}$ $> 2^{1000}$
number of vertical asymptotes of $f(x) = \frac{x^2+2x+1}{x^2+4x+3} = \frac{(x+1)(x+1)}{(\cancel{x+1})(x+3)}$	$=$	number of vertical asymptotes of $g(x) = \frac{x+1}{x+3}$ (1)
number of vertical asymptotes of $y = \ln(x)$	$=$	number of horizontal asymptotes of $y = e^x$ (1)
 number of points on the graph of $x^2 + y^2 = 4$ at which the tangent line has slope $m = 0$	$=$	number of points on the graph of $x^2 + y^2 = 4$ at which the tangent line to has slope $m = -1$ 
$0 = \lim_{x\to\infty} \frac{x^4}{x^{44}}$	$=$	$\lim_{x\to-\infty} \frac{1}{x} = 0$
$33 \cdot 23 = \sum_{k=1}^{33} 23$	$=$	$\sum_{k=3}^{35} 23 = 33 \cdot 23$
$\sum_{k=1}^{33} 23k$	$<$	$\sum_{k=3}^{35} 23k$
$\frac{\pi}{4} = \tan^{-1}(1)$	$<$	$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$ 
$f'(x) = 2x$ $f''(x) = 2$ $f''(0) = 2$ $f''(0)$ where $f(x) = x^2$	$>$	$g''(0)$ where $g(x) = x^3$ $g'(x) = 3x^2$ $g''(x) = 6x$ $g''(0) = 0$
number of inflection points for $f(x) = x^3$ $= 1$	$<$	number of inflection points for of $f(x) = \sin(x)$ $= \infty$

2. Warm up. (Quick answer.)

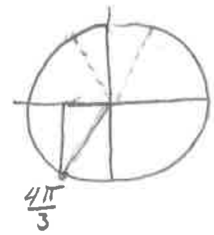
(a) $\lim_{x \to 0} \frac{\sin(x)}{x} = \boxed{1}$ (standard fact, or use L'Hôpital)

(b) $\lim_{x \to \infty} \frac{\sin(x)}{x} = \boxed{0}$ $\frac{\text{value between } -1 \text{ and } 1}{\infty}$

(c) $\lim_{x \to \pi/2} \frac{\sin(x)}{x} = \frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi/2} = \boxed{\frac{2}{\pi}}$

(d) $\frac{d}{dx}\left[\frac{\sin(x)}{x}\right] = \frac{\cos(x)\,x - \sin(x)\,(1)}{x^2} = \boxed{\frac{x\cos x - \sin x}{x^2}}$

(e) $\sin\left(\frac{4\pi}{3}\right) = \boxed{-\frac{\sqrt{3}}{2}}$



(f) $\frac{d}{dx}\left[\int_1^x \frac{\sin(t)}{t}\,dt\right] = \boxed{\frac{\sin x}{x}}$ (F.T.C. I)

(g) $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \frac{d}{dx}\left[\sin x\right] = \boxed{\cos x}$ (definition of derivative)

(h) $\frac{d}{dx}\left[x^3 \sin(x)\right] = \boxed{3x^2 \sin x + x^3 \cos x}$ (product rule)

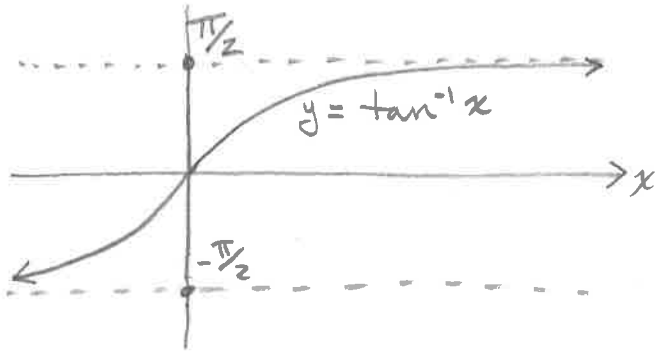
(i) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\sin^{-1}(x) + C}$ (standard formula)

(j) $\frac{d}{dx}\left[\sin^{-1}(x)\right] = \boxed{\frac{1}{\sqrt{1-x^2}}}$ (standard formula)

(k) $\sin(\tan^{-1} x) = \frac{\text{opp}}{\text{hyp}} = \boxed{\frac{x}{\sqrt{1+x^2}}}$

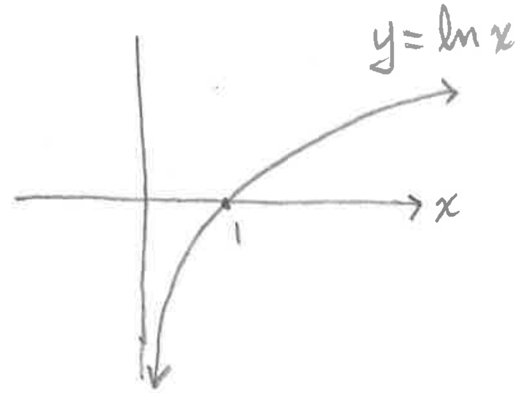
(l) $\tan^{-1}(\sqrt{3}) = \boxed{\frac{\pi}{3}}$

(m) $\lim_{x\to\infty} \tan^{-1}(x) = \boxed{\frac{\pi}{2}}$



(n) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = \boxed{-\frac{1}{2}}$

(o) $\lim_{x\to 0^+} \ln(x) = \boxed{-\infty}$



(p) $\lim_{x\to 1} \ln(x) = \ln(1) = \boxed{0}$

(q) $\int_1^e \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^e = \ln e - \ln 1 = 1 - 0 = \boxed{1}$

(r) If $f(x) = e^{x+3}$, then $f^{-1}(x) = \boxed{\ln(x) - 3}$

$y = e^{x+3}$
$x = e^{y+3}$
$\ln x = \ln e^{y+3}$
$\ln x = y + 3$
$y = \ln x - 3$

(s) $\frac{d}{dx}\left[\sqrt[5]{x^7}\right] = \frac{d}{dx}\left[x^{\frac{7}{5}}\right] = \frac{7}{5}x^{\frac{7}{5}-1} = \frac{7}{5}x^{\frac{2}{5}} = \boxed{\frac{7\sqrt[5]{x^2}}{5}}$

(t) $\int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = \boxed{9}$

3. Evaluate each limit, or explain why it does not exist.

(a) $\lim_{h\to 0} \dfrac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \dfrac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \dfrac{\frac{-h}{4+2h}}{\frac{h}{1}} = \lim_{h\to 0} \dfrac{-h}{4+2h}\,\dfrac{1}{h}$

$= \lim_{h\to 0} \dfrac{-1}{4+2h} = \dfrac{-1}{4+2\cdot 0} = \boxed{-\frac{1}{4}}$

(b) $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5} = \lim_{x\to -5} \dfrac{(x-2)(x+5)}{x+5} = \lim_{x\to -5} x-2 = \boxed{-7}$

4. Use the limit definition of the derivative to find the derivative of the function $f(x) = 3x^2$.

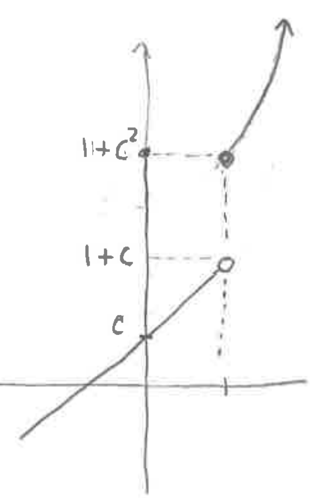
$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{3(x+h)^2 - 3x^2}{h}$

$= \lim_{h\to 0} \dfrac{3(x^2+2xh+h^2)-3x^2}{h} = \lim_{h\to 0} \dfrac{6xh+3h^2}{h}$

$= \lim_{h\to 0} \dfrac{h(6x+3h)}{h} = \lim_{h\to 0}(6x+3h) = 6x+0 = \boxed{6x}$

5. Let the function f be defined as follows. $f(x) = \begin{cases} x+c & x<1 \\ x^2+c^2 & 1 \le x \end{cases}$

For what values of c, if any, is f continuous at $x = 1$?



The only possible break is at $x=1$

For the graphs to match, we must have $1+c^2 = 1+c$

$c^2 = c$

$c^2 - c = 0$

$c(c-1) = 0$

$c=0 \qquad c=1$

Answer: Either $c=0$ or $c=1$ will make the function continuous.

6. Find the indicated derivatives.

(a) $\frac{d}{dx}\left[\sqrt{x^3+x^2+1}\right] = \frac{d}{dx}\left[(x^3+x^2+1)^{\frac{1}{2}}\right] = \frac{1}{2}(x^3+x^2+1)^{-\frac{1}{2}}(3x^2+2x)$

$$= \boxed{\frac{3x^2+2x}{2\sqrt{x^3+x^2+1}}}$$

(b) $\frac{d}{dx}\left[x\ln(\sec(3x))\right] = (1)\ln\left(\sec(3x)\right) + x\,\frac{\sec(3x)\tan(3x)\cdot 3}{\sec(3x)}$

$$= \boxed{\ln\left(\sec(3x)\right) + 3x\tan(3x)}$$

7. The equation $x^3 + y^3 + 2xy = 4$ determines a curve in the xy-plane. Find the slope of the tangent line to the curve at the point $(1,1)$.

$$\frac{d}{dx}\left[x^3+y^3+2xy\right] = \frac{d}{dx}[4]$$

$$3x^2 + 3y^2y' + 2y + 2xy' = 0$$

$$3y^2y' + 2xy' = -3x^2 - 2y$$

$$y'(3y^2+2x) = -3x^2 - 2y$$

$$y' = \frac{-3x^2-2y}{3y^2+2x}$$

$$y'\Big|_{(x,y)=(1,1)} = \frac{-3\cdot 1^2 - 2\cdot 1}{3\cdot 1^2 + 2\cdot 1} = \frac{-5}{5} = \boxed{-1}$$

8. Find the following indefinite integrals.

(a) $\int \frac{\cos(x)}{2\sin(x)+1}\,dx = \frac{1}{2}\int \frac{1}{2\sin(x)+1}\,2\cos(x)\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C$

$= \boxed{\frac{1}{2}\ln\left|2\sin(x)+1\right|+C}$

$u = 2\sin(x)+1$

$\frac{du}{dx} = 2\cos(x)$

$du = 2\cos(x)\,dx$

(b) $\int (x^3-4x+2)^3(3x^2-4)\,dx = \int u^3\,du = \frac{u^4}{4}+C = \boxed{\frac{(x^3-4x+2)^4}{4}+C}$

$u = x^3-4x+2$

$\frac{du}{dx} = 3x^2-4$

$du = (3x^2-4)\,dx$

9. Find the following definite integrals.

(a) $\int_{-1}^{1} x\sqrt{x^2+3}\,dx = \frac{1}{2}\int_{-1}^{1}\sqrt{x^2+3}\;2x\,dx = \frac{1}{2}\int_{(-1)^2+3}^{(1)^2+3} u^{1/2}\,du$

$u = x^2+3$

$du = 2x\,dx$

$= \int_4^4 u^{1/2}\,du = \left[\frac{u^{3/2}}{3/2}\right]_4^4 = \left[\frac{2\sqrt{u}^{\,3}}{3}\right]_4^4 = \boxed{0}$

(b) $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big]_0^{\sqrt{3}} = \tan^{-1}(\sqrt{3}) - \tan^{-1}(0)$

$= \frac{\pi}{3} - 0 = \boxed{\frac{\pi}{3}}$

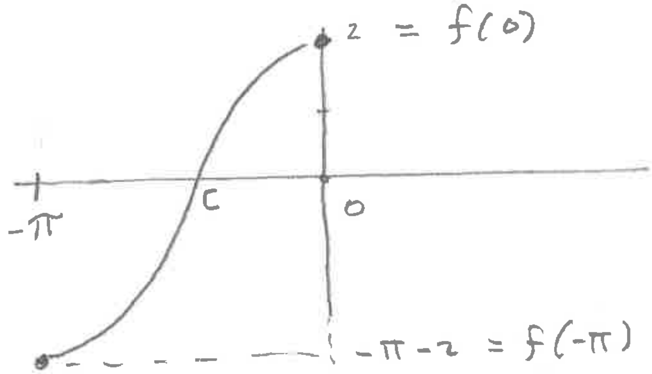
10. Show that the equation $x + 2\cos x = 0$ has at least one real solution.
Explain which major theorem in Calculus you used to answer this question and how it helped you solve the problem.

$f(x) = x + 2\cos x$

Note $f(0) = 2$ and

$f(-\pi) = -\pi - 2\cos(-\pi)$

$= -\pi - 2$



Since $f(-\pi) < 0 < f(0)$, and $f(x)$ is continuous the intermediate value theorem guarantees a number c between $-\pi$ and 0 for which $f(c) = 0$. This is a solution for $x^2 + 2\cos x = 0$.

11. A particle is traveling along the curve $y^2 - x^3 = 1$.
As it reaches the point $(2, 3)$, the y-coordinate is increasing at the rate of 4 cm/s.
How fast is the x-coordinate changing at that instant?

Know $\frac{dy}{dt} = 4$ at $(x,y) = (2,3)$

Want $\frac{dx}{dt}$ at that instant.

$$y^2 - x^3 = 1$$

$$\frac{d}{dt}\left[y^2 - x^3\right] = \frac{d}{dt}[1]$$

$$2y\frac{dy}{dt} - 3x^2\frac{dx}{dt} = 0$$

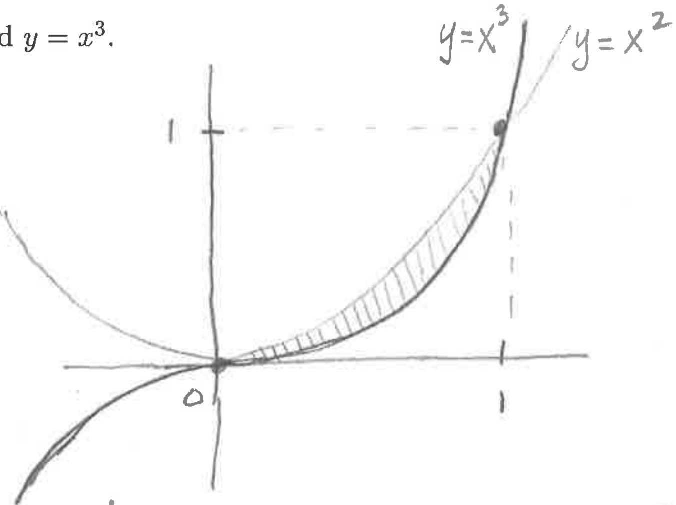
$$2y\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2y}{3x^2}\frac{dy}{dt}$$

$$= \frac{2\cdot 3}{3\cdot 2^2}\cdot 4$$

$$= \boxed{2 \text{ cm/sec}}$$

12. Find the area enclosed between the graphs of $y = x^2$ and $y = x^3$.



These curves intersect where

$x^2 = x^3$

$x^2 - x^3 = 0$

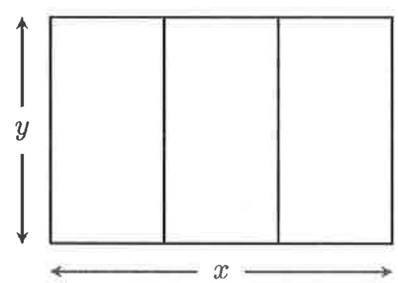
$x^2(1-x) = 0$

0   1 $\Rightarrow$ We need to find area of shaded region on right.

$$A = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1^3}{3} - \frac{1^4}{4}\right) - \left(\frac{0^3}{3} - \frac{0^4}{4}\right)$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \boxed{\frac{1}{12} \text{ square unit}}$$

13. You have a 200 feet of chain link fence to enclose <u>three</u> rectangular pens, as illustrated. What dimensions (i.e. x feet by y feet) yield the greatest total enclosed area?



Note $2x + 4y = 200$

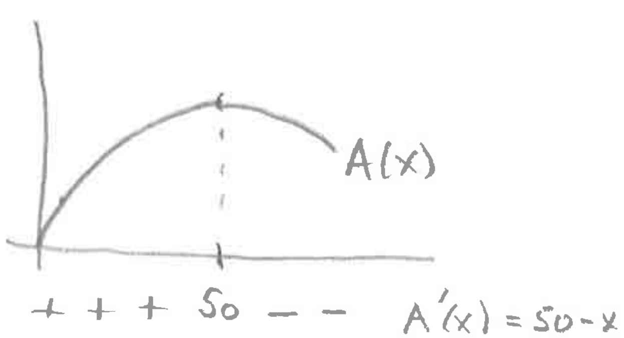
so $4y = 200 - 2x$

$y = 50 - \frac{1}{2}x$

Want to maximize area $= xy = x\left(50 - \frac{1}{2}x\right) = 50x - \frac{1}{2}x^2$

$A(x) = 50x - \frac{1}{2}x^2 \leftarrow$ maximize this on $[0, 100]$

$A'(x) = 50 - x = 0 \leftarrow x = 50$ is critical point



<u>Answer</u> Area is maximized when

$x = \boxed{50}$

$y = 50 - \frac{1}{2} \cdot 50 = \boxed{25}$