VCU

MATH 200

Calculus I

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Test 2

✍

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Name: ______________________________

Score: ____________

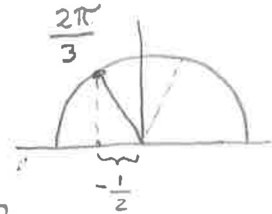
Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

1. (20 points) Warmup: short answer.

(a) $\tan^{-1}(1) = \boxed{\frac{\pi}{4}}$

(b) $\cos^{-1}(-1/2) = \boxed{\frac{2\pi}{3}}$



(c) $e^{1+\cos(\pi)} = e^{1-1} = e^{0} = \boxed{1}$

(d) $\ln(25) + 2\ln\left(\frac{e}{5}\right) = \ln(25) + \ln\left(\left(\frac{e}{5}\right)^2\right) =$

$\ln(25) + \ln\left(\frac{e^2}{25}\right) = \ln(25) + \ln(e^2) - \ln(25) = \boxed{2}$

(e) If $f(x) = \ln(x)$, then $f^{-1}(x) = \boxed{e^x}$

(f) If $f(x) = \ln(x)$, then $f'(x) = \boxed{\frac{1}{x}}$

(g) $\lim_{h \to 0} \frac{\ln(8+h) - \ln(8)}{h} = \boxed{\frac{1}{8}}$ ← Derivative of $\ln(x)$ at $x = 8$

(h) $\frac{d}{dx}\left[\sin(x^{10})\right] = \boxed{\cos(x^{10})\, 10x^9}$

(i) $\frac{d}{dx}\left[\frac{1}{x} + \tan(x)\right] = \boxed{-\frac{1}{x^2} + \sec^2(x)}$

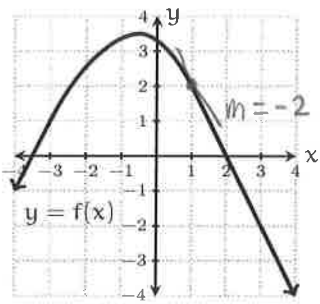
(j) $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \boxed{\frac{1}{1+x^2}}$

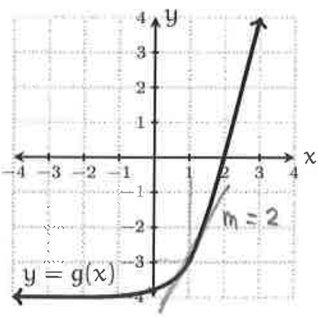
2. (5 points) Suppose $f(x)$ is the number of gallons of fuel in a rocket when it is x miles above the Earth's surface. Explain, in non-mathematical terms, what the statement $f'(20) = -8$ means.

$f'(20) = \left(\begin{array}{l}\text{rate of change of} \\ f(x) = \text{fuel when } x = 20\end{array}\right) = -8 \; \frac{\text{gallons}}{\text{mile}}$

When the rocket is 20 miles high it is using fuel at a rate of 8 gallons per mile

3. (5 points) Two functions $f(x)$ and $g(x)$ are graphed below. Let $h(x) = f(x)g(x)$. Estimate $h'(1)$. Show your work.



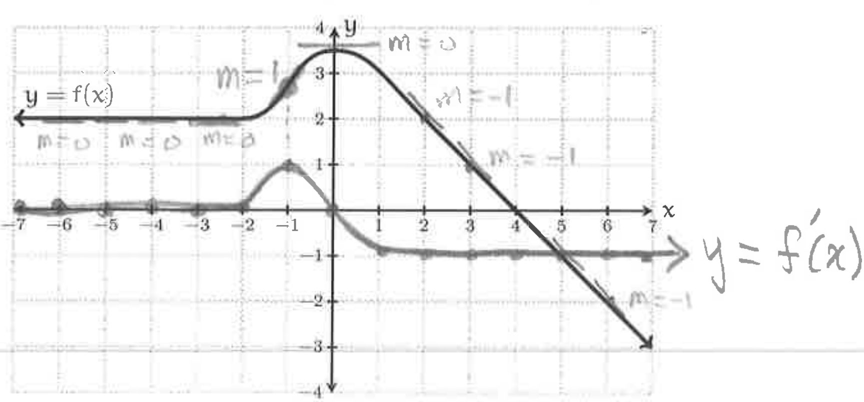


$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(1) = f'(1)g(1) + f(1)g'(1)$

$= (-2)(-3) + (2)(2) = 6 + 4 = \boxed{10}$

4. (5 points) A function $f(x)$ is graphed below. Using the same coordinate axis, sketch the graph of the derivative $f'(x)$.



5. (20 points) Find the following derivatives.

(a) $\frac{d}{dx}[\ln(\tan(x))] = \frac{1}{\tan(x)} \sec^2(x) = \boxed{\frac{\sec^2(x)}{\tan(x)}}$

$= \frac{\frac{1}{\cos^2(x)}}{\frac{\sin(x)}{\cos(x)}} = \boxed{\frac{1}{\cos(x)\sin(x)}}$

(b) $\frac{d}{dx}\left[\sqrt{\frac{x^2+5}{e^x}}\right] = \frac{d}{dx}\left[\left(\frac{x^2+5}{e^x}\right)^{\frac{1}{2}}\right] =$

$\boxed{\frac{1}{2}\left(\frac{x^2+5}{e^x}\right)^{-\frac{1}{2}} \frac{2xe^x - (x^2+5)e^x}{(e^x)^2}}$

(c) $\frac{d}{dx}[\sqrt[3]{x}\sin(x)] = \frac{d}{dx}\left[x^{\frac{1}{3}}\sin(x)\right]$

$= \frac{1}{3}x^{-2/3}\sin(x) + x^{\frac{1}{3}}\cos(x)$

$= \boxed{\frac{\sin(x)}{3\sqrt[3]{x}^2} + \sqrt[3]{x}\cos(x)}$

(d) $\frac{d}{dx}[e^{1+e^{-x}}] =$

$= e^{1+e^{-x}} \frac{d}{dx}[1+e^{-x}]$

$= e^{1+e^{-x}}(0 + e^{-x}(-1))$

$= \boxed{-e^{-x}e^{1+e^{-x}}}$

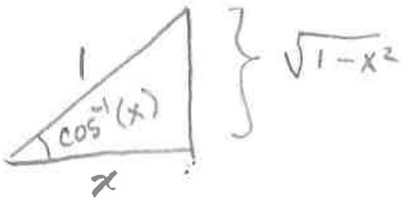
6 (10 points) Use logarithmic differentiation to find the derivative of the function $y = (x^2 + 1)^x$.

$$\ln(y) = \ln(x^2+1)^x$$
$$\ln(y) = x\ln(x^2+1)$$
$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[x\ln(x^2+1)\right]$$
$$\frac{y'}{y} = (1)\ln(x^2+1) + x\frac{2x}{x^2+1}$$
$$y' = y\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)$$
$$\boxed{y' = (x^2+1)^x\left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)}$$

7 (10 points) Use the limit definition of the derivative to find the derivative of the function $f(x) = x^2 + 1$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)^2+1-(x^2+1)}{h}$$
$$= \lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h}$$
$$= \lim_{h\to 0}\frac{2xh+h^2}{h} =$$
$$= \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0}(2x+h)$$
$$= 2x + 0 = \boxed{2x}$$

8. (5 points) Simplify: $\tan\left(\cos^{-1}(x)\right) = \dfrac{\text{OPP}}{\text{ADJ}} = \boxed{\dfrac{\sqrt{1-x^2}}{x}}$



9. (10 points) Suppose $f(x) = \dfrac{5}{x}$.

Find the **equation** of the line tangent to the graph of $y = f(x)$ at the point $(5, f(5))$.

$f'(x) = -\dfrac{5}{x^2}$

Slope $f'(5) = -\dfrac{5}{5^2} = -\dfrac{5}{25} = -\dfrac{1}{5}$

Point $(5, f(5)) = (5, \frac{5}{5}) = (5, 1)$

By point/slope formula:

$y - y_0 = m(x - x_0)$

$y - 1 = -\frac{1}{5}(x - 5)$

$y - 1 = -\frac{1}{5}x + 1$

$$\boxed{y = -\frac{1}{5}x + 2}$$

10. (10 points) This question concerns the equation $x^3 + y^3 = 4xy$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x^3+y^3\right] = \frac{d}{dx}\left[4xy\right]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 4y + 4x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 4x\frac{dy}{dx} = 4y - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 4x\right) = 4y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{4y-3x^2}{3y^2-4x}}$$

(b) Use your answer from part (a) to find the slope of the tangent line to the graph of $x^3 + y^3 = 4xy$ at the point $(2, 2)$.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = \frac{4(2) - 3\cdot 2^2}{3\cdot 2^2 - 4\cdot 2}$$

$$= \frac{8-12}{12-8} = \frac{-4}{4}$$

$$= \boxed{-1}$$