VCU

MATH 200

Calculus I

R. Hammack

Test 3

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Name: Richard

Score: ______________

Directions. Answer the questions in the provided space. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Please put all phones away.

1. (25 points) Find the indefinite integrals.

(a) $\int (x^3 + 3x + 5)\, dx = \boxed{\frac{x^4}{4} + \frac{3}{2}x^2 + 5x + C}$

(b) $\int \frac{1}{\sqrt{x}}\, dx = \int x^{-1/2}\, dx = \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C$

$= \frac{1}{1/2} x^{1/2} + C = \boxed{2\sqrt{x} + C}$

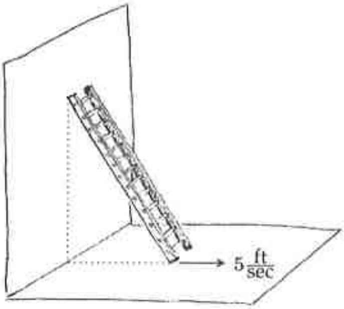
(c) $\int \frac{e^x + 1}{e^x}\, dx = \int \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx = \int (1 + e^{-x})\, dx$

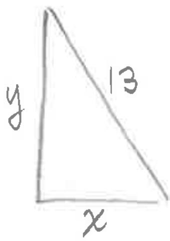
$= \boxed{x - e^{-x} + C}$

(d) $\int \frac{1}{1 + x^2}\, dx = \boxed{\tan^{-1}(x) + C}$

(e) $\int (\sec^2(x) + 3\sin(x))\, dx = \boxed{\tan(x) - 3\cos(x) + C}$

2. (15 pts.) A 13-foot ladder leans against a wall, as shown. Its base slides away from the wall at a rate of 5 feet per second. How fast is the top of the ladder sliding down the wall when its base is 12 feet from the wall?





Know $\frac{dx}{dt} = 5$

Want $\frac{dy}{dt}$

(when $x = 12$)

$$x^2 + y^2 = 13^2$$

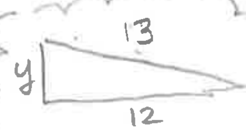
$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[169\right]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2x \cdot 5 + 2y\frac{dy}{dt} = 0$$

$$5x + y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -5x$$



$y^2 + 12^2 = 13^2$

$y^2 = 13^2 - 12^2$

$= 169 - 144$

$= 25$

$\Rightarrow y = \sqrt{25} = 5$

$$\frac{dy}{dt} = -\frac{5x}{y} = -\frac{5 \cdot 12}{5} = \boxed{-12\,\frac{ft}{sec}}$$

3. (15 pts.) Consider the function $f(x) = 5x^4 + 20x^3 + 10$.

(a) Find the critical points of $f(x)$.

$$f'(x) = 20x^3 + 60x^2$$
$$= 20x^2(x+3) = 0$$

$$\boxed{x=0 \qquad x=-3}$$

(b) Find the intervals on which $f(x)$ increases/decreases

-3 $\quad$ 0

$- - - \; | \; + + + + \; | \; + + + \quad f'(x)$

$\boxed{\begin{array}{l} f(x) \text{ increasing on } (-3, \infty) \\ f(x) \text{ decreasing on } (-\infty, -3) \end{array}}$

(c) State the locations of the local minima of $f(x)$ (if any).

$\boxed{\text{local min at } x=-3}$

(d) State the locations of the local maxima of $f(x)$ (if any).

$\boxed{\text{none}}$

(e) State the interval(s) on which $f(x)$ is concave down.

$$f''(x) = 60x^2 + 120x = 60x(x+2)$$

$x=0 \qquad x=-2$

-2 $\quad$ 0

$+ + + \; | \; - - - - \; | \; + + \quad f''(x)$

$\boxed{f(x) \text{ concave down on } (-2, 0)}$

4. (15 pts.) Use L'Hôpital's rule to find the limits.

(a) $\lim_{x\to 0} \frac{e^{3x}-1}{x^2+3x} = \lim_{x\to 0} \frac{3e^{3x}}{2x+3} = \frac{3e^{3\cdot 0}}{2\cdot 0+3}$

form $\frac{0}{0}$

$= \frac{3\cdot 1}{3} = \boxed{1}$

[apply L'Hôpital's Rule]

(b) $\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)\sec(x) = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{\frac{1}{\sec(x)}}$

form $0\cdot\infty$

$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{\cos(x)} = \lim_{x\to\frac{\pi}{2}} \frac{-1}{-\sin(x)}$

form $\frac{0}{0}$

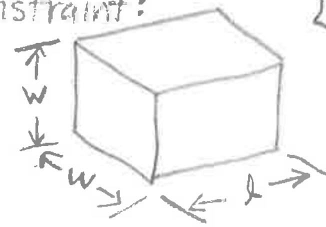
Apply L'Hopital

$= \frac{-1}{-\sin\left(\frac{\pi}{2}\right)}$

$= \frac{-1}{-1} = \boxed{1}$

5 (15 pts.) USPS rules say the length plus girth of a package cannot exceed 108 inches. (Girth $= 2 \cdot \text{width} + 2 \cdot \text{height}$, as illustrated.) You must mail a package whose width and height are equal, and with the greatest possible volume. Find the dimensions of the package.

Constraint:



$$\underbrace{4w}_{\text{girth}} + \underbrace{l}_{\text{length}} = 108$$

$$\Rightarrow \boxed{l = 108 - 4w}$$

Maximize:

$$\text{Volume} = lww$$
$$= (108-4w)w^2$$

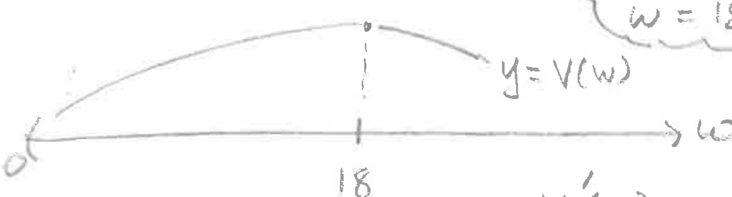
$$V(w) = 108w^2 - 4w^3$$

Find w that maximizes this on $(0, 108)$

$$V'(w) = 216w - 12w^2$$
$$= 12w(18-w) = 0$$

$w = 0$ $\qquad$ $w = 18$

Only critical point in interval is $w = 18$



$+ + + + + \mid - - -$ $V'(w)$

$l = 108 - 4 \cdot 18$
$= 108 - 72$
$=$

Answer Maximum volume at $\boxed{\begin{array}{l} w = 18 \\ l = 36 \end{array}}$

6. (15 pts.) A ball, tossed straight up, has a constant acceleration of -32 feet per second per second. At time $t = 0$ its velocity is $v(0) = 20$ feet per second, and its position is $s(0) = 5$ feet. Find the position function $s(t)$.

$$V(t) = \int a(t)\,dt = \int -32\,dt$$

$$V(t) = -32t + C$$

$$20 = V(0) = -32 \cdot 0 + C \implies \boxed{C = 20}$$

Therefore $\boxed{V(t) = -32t + 20}$

Then $s(t) = \int v(t)\,dt = \int (-32t + 20)\,dt$

$$= -32\frac{t^2}{2} + 20t + C$$

$$S(t) = -16t^2 + 20t + C$$

$$5 = S(0) = -16 \cdot 0^2 + 20 \cdot 0 + C$$

$$\boxed{5 = C}$$

Answer: $\boxed{S(t) = -16t^2 + 20t + 5}$