VCU

MATH 200

CALCULUS I

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TEST 2

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Name: Richard

Score: 100

Directions. Answer the questions in the provided space. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Please put all phones away.

1. (20 points) Warmup: short answer.

(a) If $f(x) = \tan^{-1}(x) + \frac{1}{x}$, then $f'(x) = \boxed{\frac{1}{1+x^2} - \frac{1}{x^2}}$

(b) If $f(x) = \tan(x) + e^x$, then $f'(x) = \boxed{\sec^2(x) + e^x}$

(c) If $f(x) = \sqrt[5]{x^6}$, then $f'(x) = \frac{6}{5}x^{\frac{1}{5}} = \boxed{\frac{6\sqrt[5]{x}}{5}}$

$= x^{\frac{6}{5}}$

(d) If $f(x) = \frac{1}{2}\sin(x) + e$, then $f'(x) = \boxed{\frac{1}{2}\cos(x)}$

(e) If $f(x) = e^{\pi x}$, then $f'(x) = \boxed{\pi e^{\pi x}}$

(f) If $f(x) = e^{\pi x}$, then $f'(0) = \pi e^{\pi \cdot 0} = \pi e^0 = \boxed{\pi}$

(g) $\lim\limits_{h \to 0} \dfrac{e^{x+h} - e^x}{h} = \boxed{e^x}$

(h) $\dfrac{d}{dx}\left[\sin^{-1}(\pi x)\right] = \dfrac{1}{\sqrt{1-(\pi x)^2}}\pi = \boxed{\dfrac{\pi}{\sqrt{1-\pi^2 x^2}}}$

(i) $\dfrac{d}{dx}\left[\ln(\sin(x))\right] = \dfrac{\cos(x)}{\sin(x)} = \boxed{\cot(x)}$

(j) $\dfrac{d}{dx}\left[\dfrac{1}{x^2+3x}\right] = \dfrac{d}{dx}\left[(x^2+3x)^{-1}\right] = -(x^2+3x)^{-2}(2x+3)$

$= \boxed{\dfrac{-(2x+3)}{(x^2+3x)^2}}$

2. (5 points) Find the equation of the tangent line to the graph of $y = \sin(x)$ at the point where $x = 2\pi$.

Slope at x is $y' = \cos(x)$

For $x = 2\pi$, $m = \cos(2\pi) = 1$

Point on tangent is $(2\pi, \sin(2\pi)) = (2\pi, 0)$

By point-slope formula

$y - y_0 = m(x - x_0)$

$y - 0 = 1(x - 2\pi)$

$\boxed{y = x - 2\pi}$

3. (5 points) Information about a function $f(x)$ and its derivative is given in the table below.

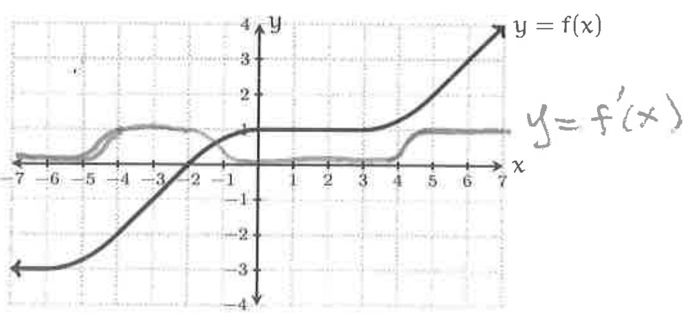
x	0	1	2	3	4	5
$f(x)$	0	−3	−2	3	10	25
$f'(x)$	−1	−7	−5	5	20	30

Suppose $h(x) = f(x^2)$. Find $h'(2)$. Show your work.

$h'(x) = f'(x^2)2x$

$h'(2) = f'(2^2)2 \cdot 2 = f'(4) \cdot 4 = 20 \cdot 4 = \boxed{80}$

4. (5 points) A function $f(x)$ is graphed below. Using the same coordinate axis, sketch the graph of the derivative $f'(x)$.



5. (20 points) Find the following derivatives.

(a) $\frac{d}{dx}\left[\sec\left(e^{3x+1}\right)\right] =$

$$\sec\left(e^{3x+1}\right)\tan\left(e^{3x+1}\right)e^{3x+1}\cdot 3$$

$$= \boxed{3e^{3x+1}\sec\left(e^{3x+1}\right)\tan\left(e^{3x+1}\right)}$$

(b) $\frac{d}{dx}\left[\ln\left(x^{10} - 4x^2 + 1\right)\right] =$ $\boxed{\dfrac{10x^9 - 8x}{x^{10} - 4x^2 + 1}}$

(c) $\frac{d}{dx}\left[\tan(x^5) + \tan^5(x)\right] =$

$$\boxed{\sec^2(x^5)\,5x^4 + 5\tan^4(x)\sec^2(x)}$$

(d) $\frac{d}{dx}\left[\frac{x^3\ln(x)}{x^3+1}\right] = \dfrac{\left(3x^2\ln(x) + x^3\frac{1}{x}\right)(x^3+1) - x^3\ln(x)\,3x^2}{(x^3+1)^2}$

$$= \boxed{\dfrac{\left(3x^2\ln(x) + x^2\right)(x^3+1) - 3x^5\ln(x)}{(x^3+1)^2}}$$

Use logarithmic differentiation

6 (10 points) Find the derivative of $y = x^{\ln(x)}$.

$$\ln(y) = \ln\left(x^{\ln(x)}\right)$$

$$\ln(y) = \ln(x)\ln(x)$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\ln(x)\ln(x)\right]$$

$$\frac{y'}{y} = \frac{1}{x}\ln(x) + \ln(x)\frac{1}{x}$$

$$y' = y\left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x}\right)$$

$$y' = x^{\ln(x)}\, 2\,\frac{\ln(x)}{x} = \boxed{\frac{2\ln(x)\, x^{\ln(x)}}{x}}$$

7 (10 points) Consider $f(x) = x^3 - 27x + 5$. Find all x for which the tangent to $y = f(x)$ at the point $(x, f(x))$ is horizontal.

$$f'(x) = 3x^2 - 27 = 0$$

horizontal tangent line means zero slope

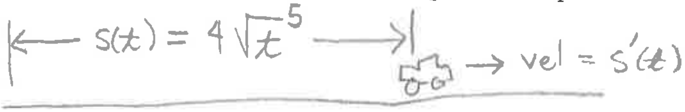
$$3(x^2 - 9) = 0$$

$$3(x-3)(x+3) = 0$$

$x = 3$ $\quad$ $x = -3$

Answer: $\boxed{\text{Tangent horizontal at } x = 3 \text{ and } x = -3}$

8. (15 points) An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 4\sqrt{t}^5$ feet. What is its velocity is when its acceleration is 30 feet per second per second?



position: $s(t) = 4t^{\frac{5}{2}}$

velocity: $v(t) = s'(t) = 4\,\frac{5}{2}\,t^{\frac{3}{2}} = 10\sqrt{t}^{\,3}$

acceleration $a(t) = v'(t) = 10\,\frac{3}{2}\,t^{\frac{3}{2}-1} = 15\sqrt{t}$

To find when acceleration is 30, we solve the equation

$$a(t) = 30$$

$$15\sqrt{t} = 30$$

$$\sqrt{t} = 2$$

$$t = 4 \text{ sec}$$

Thus acceleration is 30 ft/sec^2 when $t = 4$, so at this time the velocity is $v(4) = 10\sqrt{4}^{\,3} = 10 \cdot 2^3 = 10 \cdot 8$

$$= \boxed{80 \text{ ft/sec}}$$

$$x\, y^{\frac{2}{3}} + y = 12$$

9. (10 points) This question concerns the equation $x\sqrt[3]{y}^2 + y = 12$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x y^{\frac{2}{3}} + y\right] = \frac{d}{dx}\left[12\right]$$

$$(1)\, y^{2/3} + x\,\frac{2}{3}\, y^{-\frac{1}{3}}\, y' + y' = 0$$

$$\frac{2xy'}{3y^{1/3}} + y' = -y^{2/3}$$

$$y'\left(\frac{2x}{3\sqrt[3]{y}} + 1\right) = -\sqrt[3]{y}^2$$

$$\boxed{y' = \frac{-\sqrt[3]{y}^2}{\frac{2x}{3\sqrt[3]{y}} + 1}}$$

(b) Use your answer from part (a) to find the slope of the tangent line to the graph of $x\sqrt[3]{y}^2 + y = 12$ at the point $(1, 8)$.

$$y'\Big|_{(x,y)=(1,8)} = \frac{-\sqrt[3]{8}^2}{\frac{2\cdot 1}{3\sqrt[3]{8}} + 1}$$

$$= \frac{-2^2}{\frac{2}{6} + 1} = \frac{-4}{8/6}$$

$$= -\frac{24}{8} = \boxed{-3}$$