VCU

MATH 200

Calculus I

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Test 2

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Score: 100

Directions. Answer the questions in the provided space. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Please put all phones away.

1. (20 points) Warmup: short answer.

(a) If $f(x) = \tan(x) + \ln(x)$, then $f'(x) = \boxed{\sec^2(x) + \frac{1}{x}}$

(b) If $f(x) = \sin^{-1}(x) + e^x$, then $f'(x) = \boxed{\frac{1}{\sqrt{1-x^2}} + e^x}$

(c) If $f(x) = \sqrt[3]{x^5}$, then $f'(x) = \frac{5}{3}x^{\frac{5}{3}-1} = \frac{5}{3}x^{\frac{2}{3}} = \boxed{\frac{5}{3}\sqrt[3]{x}^2}$

$= x^{\frac{5}{3}}$

(d) If $f(x) = \frac{1}{2}\sin(x) + e$, then $f'(x) = \boxed{\frac{1}{2}\cos(x)}$

(e) If $f(x) = e^{-x}$, then $f'(x) = \boxed{-e^{-x}}$

(f) If $f(x) = e^{-x}$, then $f'(\ln(2)) = -e^{-\ln(2)} = -\frac{1}{e^{\ln(2)}} = \boxed{-\frac{1}{2}}$

(g) $\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \boxed{e^x}$ (definition of $\frac{d}{dx}[e^x]$)

(h) $\frac{d}{dx}\left[\tan^{-1}(\pi x)\right] = \frac{1}{1+(\pi x)^2}\pi = \boxed{\frac{\pi}{1+\pi^2 x^2}}$

(i) $\frac{d}{dx}\left[\ln(\cos(x))\right] = \frac{-\sin(x)}{\cos(x)} = \boxed{-\tan(x)}$

(j) $\frac{d}{dx}\left[\frac{1}{x^2+3x}\right] = \frac{d}{dx}\left[(x^2+3x)^{-1}\right] = -(x^2+3x)^{-2}(2x+3)$

$= \boxed{-\frac{2x+3}{(x^2+3x)^2}}$

2. (5 points) Find the equation of the tangent line to the graph of $y = \sin(x)$ at the point where $x = \pi$.

Slope at x is $y' = \cos(x)$

Thus $m = \cos(\pi) = -1$

Point on tangent is $(\pi, \sin(\pi)) = (\pi, 0)$

Point-slope form:

$y - 0 = -1(x - \pi)$

$$\boxed{y = -x + \pi}$$

3. (5 points) Information about a function $f(x)$ and its derivative is given in the table below.

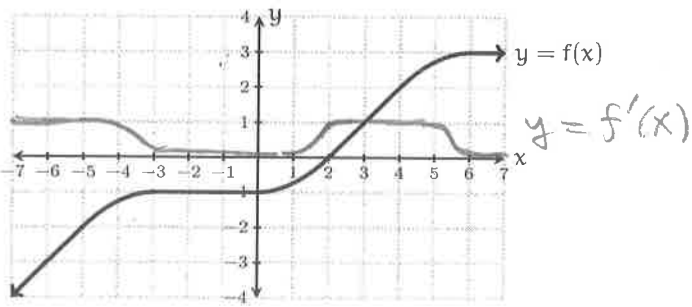
x	0	1	2	3	4	5
$f(x)$	0	−3	−2	3	10	25
$f'(x)$	−1	−7	−5	5	20	30

Suppose $h(x) = (f(x))^3$. Find $h'(2)$. Show your work.

$h'(x) = 3(f(x))^2 f'(x)$

$h'(2) = 3(f(2))^2 f'(2) = 3\cdot(-2)^2(-5) = \boxed{-60}$

4. (5 points) A function $f(x)$ is graphed below. Using the same coordinate axis, sketch the graph of the derivative $f'(x)$.



5. (20 points) Find the following derivatives.

(a) $\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right] =$ $\frac{0-\frac{1}{x^2}}{1+\frac{1}{x}} = \frac{-\frac{1}{x^2}}{\frac{x+1}{x}}$

$$= \boxed{-\frac{1}{x(x+1)}}$$

(b) $\frac{d}{dx}\left[\tan(x^5)+\tan^5(x)\right] =$

$$= \sec^2(x^5)\frac{d}{dx}[x^5] + 5\tan^4(x)\frac{d}{dx}[\tan(x)]$$

$$= \boxed{\sec^2(x^5)\,5x^4 + 5\tan^4(x)\sec^2(x)}$$

(c) $\frac{d}{dx}\left[\sec\left(e^{x^3+x}\right)\right] =$

$$\boxed{\sec\left(e^{x^3+x}\right)\tan\left(e^{x^3+x}\right)e^{x^3+x}\,(3x^2+1)}$$

(d) $\frac{d}{dx}\left[\frac{x^3\ln(x)}{x^3+1}\right] =$

$$\frac{\left(3x^2\ln(x)+x^3\frac{1}{x}\right)(x^3+1)-x^3\ln(x)\,3x^2}{(x^3+1)^2}$$

$$= \boxed{\frac{(3x^2\ln(x)+x^2)(x^3+1)-3x^5\ln(x)}{(x^3+1)^2}}$$

(Use logarithmic differentiation)

6 (10 points) Find the derivative of $y = x^{\ln(x)}$.

$$\ln(y) = \ln\left(x^{\ln(x)}\right)$$

$$\ln(y) = \ln(x)\ln(x)$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\ln(x)\ln(x)\right]$$

$$\frac{y'}{y} = \frac{1}{x}\ln(x) + \ln(x)\frac{1}{x}$$

$$y' = y\left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x}\right)$$

$$\boxed{y' = x^{\ln(x)}\, 2\,\frac{\ln(x)}{x}}$$

7 (10 points) Consider $f(x) = 2x^3 - 3x^2 - 12x + 4$. Find all x for which the tangent to $y = f(x)$ at the point $(x, f(x))$ is horizontal.

$$f'(x) = 6x^2 - 6x - 12 = 0$$

horizontal means slope is 0

$$6(x^2 - x - 2) = 0$$

$$6(x+1)(x-2) = 0$$

$$x = -1 \qquad x = 2$$

$$\boxed{\text{Answer } x = -1 \text{ and } x = 2}$$

8. (15 points) An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 4\sqrt{t}^5$ feet. What is its velocity is when its acceleration is 30 feet per second per second?

position: $S(t) = 4t^{\frac{5}{2}}$

velocity: $V(t) = S'(t) = 4\frac{5}{2}t^{\frac{3}{2}} = 10\sqrt{t}^3$

acceleration $a(t) = V'(t) = 10\frac{3}{2}t^{\frac{3}{2}-1} = 15\sqrt{t}$

To find when acceleration is 30, we solve the equation

$$a(t) = 30$$

$$15\sqrt{t} = 30$$

$$\sqrt{t} = 2$$

$$t = 4 \text{ sec}$$

Thus acceleration is 30 ft/sec^2 when $t = 4$, so at this time the velocity is $V(4) = 10\sqrt{4}^3 = 10 \cdot 2^3 = 10 \cdot 8$

$$= \boxed{80 \text{ ft/sec}}$$

$$x y^{\frac{2}{3}} + y = 12$$

9. (10 points) This question concerns the equation $x\sqrt[3]{y}^2 + y = 12$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[x y^{\frac{2}{3}} + y\right] = \frac{d}{dx}\left[12\right]$$

$$(1) y^{2/3} + x \frac{2}{3} y^{-\frac{1}{3}} y' + y' = 0$$

$$\frac{2xy'}{3y^{1/3}} + y' = -y^{2/3}$$

$$y'\left(\frac{2x}{3\sqrt[3]{y}} + 1\right) = -\sqrt[3]{y}^2$$

$$\boxed{y' = \frac{-\sqrt[3]{y}^2}{\frac{2x}{3\sqrt[3]{y}} + 1}}$$

(b) Use your answer from part (a) to find the slope of the tangent line to the graph of $x\sqrt[3]{y}^2 + y = 12$ at the point $(1, 8)$.

$$y'\Big|_{(x,y)=(1,8)} = \frac{-\sqrt[3]{8}^2}{\frac{2 \cdot 1}{3\sqrt[3]{8}} + 1}$$

$$= \frac{-2^2}{\frac{2}{6} + 1} = \frac{-4}{8/6}$$

$$= -\frac{24}{8} = \boxed{-3}$$