VCU

MATH 200

CALCULUS I

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TEST 3

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Name: Richard

Score: 100

Directions. Answer the questions in the space provided. To get full credit, please show and explain your work as appropriate. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

1. (25 points) Find the indefinite integrals.

(a) $\int \left(x^4 + \frac{1}{x} + \sqrt{2} \right) dx = \boxed{\frac{x^5}{5} + \ln|x| + \sqrt{2}x + C}$

(b) $\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = \frac{1}{-2+1} x^{-2+1} + C$

$= -x^{-1} + C = \boxed{\frac{-1}{x} + C}$

(c) $\int e^{2x}\, dx = \boxed{\frac{1}{2} e^{2x} + C}$

(d) $\int 3\sec(x)\tan(x)\, dx = \boxed{3\sec(x) + C}$

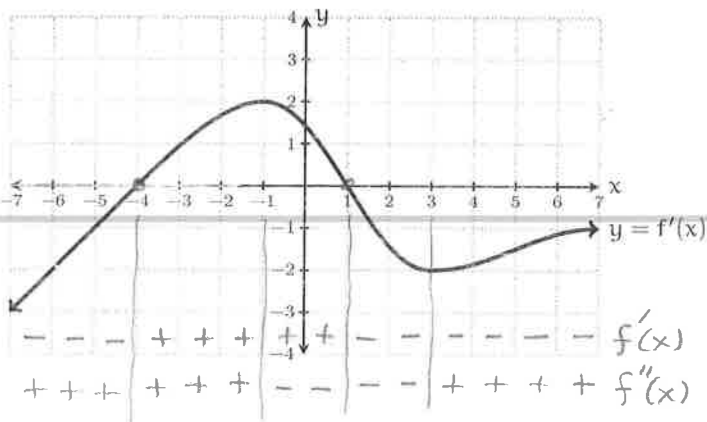
(e) If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\, dx = \boxed{\frac{f(x)}{g(x)} + C}$$

because $\frac{d}{dx}\left[\frac{f(x)}{g(x)} + C \right] =$

$$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

2. (15 pts.) The graph of the derivative $f'(x)$ of a function $f(x)$ is sketched below. Answer the following questions about the function $f(x)$.



(a) List the critical points of $f(x)$.

$\boxed{x = -4,\ 1}$ because $f'(-4) = 0$ and $f'(1) = 0$.

(b) State the interval(s) on which $f(x)$ increases.

$\boxed{(-4, 1)}$ because $f'(x) > 0$ there

(c) State the interval(s) on which $f(x)$ decreases.

$\boxed{(-\infty, -4)\ (1, \infty)}$ because $f'(x) < 0$ there

(d) State the locations of the relative extrema of $f(x)$.

local max at $x = 1$

local min at $x = -4$

(by first derivative test)

(e) State the interval(s) on which $f(x)$ is concave down.

$\boxed{(-1, 3)}$ Because $f'(x)$ decreases on this interval, and therefore $f''(x) < 0$ there.

3. (15 pts.) Consider the function $f(x) = x^2e^{-x}$.

(a) Find the critical points of $f(x)$.

$$f'(x) = 2xe^{-x} + x^2e^{-x}(-1)$$
$$= 2xe^{-x} - x^2e^{-x}$$
$$= xe^{-x}(2-x) = 0$$

$\boxed{x = 0 \qquad x = 2}$

(b) Find the intervals on which $f(x)$ increases.

0 2

$- - - \mid + + + + \mid - - - - \quad f'(x) = xe^{-x}(2-x)$

$\boxed{f(x) \text{ increases on } (0, 2)}$

because its derivative is positive there.

(c) Find the intervals on which $f(x)$ decreases.

$\boxed{(-\infty, 0) \text{ and } (2, \infty)}$

(d) State the locations of the local maxima of $f(x)$.

$\boxed{x = 2}$

(e) State the locations of the local minima of $f(x)$.

$\boxed{x = 0}$

by first derivative test.

4. (20 pts.) Use L'Hôpital's rule to find the limits.

(a) $\lim_{x\to\pi} \frac{\sin(x)}{x^2-\pi^2} = \lim_{x\to\pi} \frac{\cos(x)}{2x} = \frac{\cos(\pi)}{2\pi}$

$= \boxed{-\frac{1}{2\pi}}$

form $\frac{0}{0}$

(b) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)}$

$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$

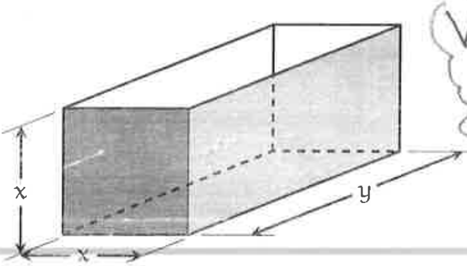
$= \lim_{x\to\infty} e^{\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}}$

form $\frac{0}{0}$

$= \lim_{x\to\infty} e^{\frac{\frac{-1/x^2}{1+1/x}}{-1/x^2}}$

$= \lim_{x\to\infty} e^{\frac{1}{1+1/x}} = e^{\frac{1}{1+0}} = e^1 = \boxed{e}$

5 (15 pts.) A metal box with two square ends and an open top is to contain a volume of 36 cubic inches. What dimensions x and y will minimize the total area of the metal surface?



$$V = xxy = 36$$

$$y = \frac{36}{x^2}$$

$$\text{Surface Area} = 2x^2 + 3xy$$

$$= 2x^2 + 3x\frac{36}{x^2}$$

$$A(x) = 2x^2 + \frac{108}{x}$$

Minimize area $A(x)$ on $(0, \infty)$

$$A'(x) = 4x - \frac{108}{x^2} = 0$$

$$4x = \frac{108}{x^2}$$

$$4x^3 = 108$$

$$x^3 = 27$$

$$x = \sqrt[3]{27} = 3$$

critical point

$y = A(x)$

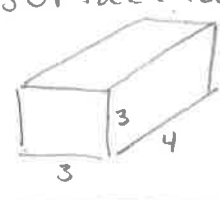
$- - - -\ 3\ + + +\ A'(x)$

Minimum area at $x = 3$

Answer: To minimize surface area:

$x = 3''$

$y = \frac{36}{3^2} = 4''$



6. (10 pts.) Suppose $f(x)$ is a function for which

$$f'(x) = \frac{3}{\sqrt[3]{x^2}} \quad \text{and} \quad f(-1) = -5. \quad \text{Find } f(x).$$

$$f(x) = \int \frac{3}{\sqrt[3]{x}^{\,2}}\,dx = \int 3x^{-\frac{2}{3}}\,dx$$

$$= 3\,\frac{1}{-\frac{2}{3}+1}\,x^{-\frac{2}{3}+1} + C$$

$$= 3\,\frac{1}{1/3}\,x^{1/3} + C$$

$$= \boxed{9\sqrt[3]{x} + C}$$

Thus $f(x) = 9\sqrt[3]{x} + C$

So $-5 = f(-1) = 9\sqrt[3]{-1} + C$

$$-5 = -9 + C$$

$$\boxed{4 = C}$$

Thus $\boxed{f(x) = 9\sqrt[3]{x} + 4}$