MATH 200

Calculus I

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Test 3

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Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

7. (10 pts.) Suppose $f(x)$ is a function for which $f'(x) = \frac{1}{x} + 3x$ and $f(1) = 5$. Find $f(x)$.

$$f(x) = \int \left(\frac{1}{x} + 3x\right) dx = \ln|x| + \frac{3}{2}x^2 + C$$

Thus we just need to find C.

$$5 = f(1) = \ln 1 + \frac{3}{2} \cdot 1^2 + C$$

$$5 = 0 + \frac{3}{2} + C$$

$$C = 5 - \frac{3}{2} = \frac{10}{2} - \frac{3}{2} = \frac{7}{2}$$

Therefore $\boxed{f(x) = \ln|x| + \frac{3}{2}x^2 + \frac{7}{2}}$

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1. (32 points) Find the indefinite integrals.

(a) $\int (5x + 3 + x^4) dx = \boxed{5\frac{x^2}{2} + 3x + \frac{x^5}{5} + C}$

(b) $\int \left(\frac{1}{x^2} + \sqrt{x}\right) dx = \int (x^{-2} + x^{\frac{1}{2}}) dx = \frac{1}{-2+1} x^{-2+1} + \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C$

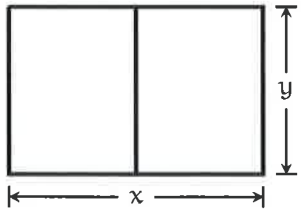
$= \frac{1}{-1} x^{-1} + \frac{1}{3/2} x^{3/2} + C = \boxed{-\frac{1}{x} + \frac{2}{3}\sqrt{x}^3 + C}$

(c) $\int \frac{6}{\sqrt{1-x^2}} dx = 6 \int \frac{1}{\sqrt{1-x^2}} dx = \boxed{6 \sin^{-1}(x) + C}$

(d) $\int 4\sin(3x)\, dx = 4 \int \sin(3x)\, dx = 4\left(-\frac{1}{3}\cos(3x)\right) + C$

$= \boxed{-\frac{4}{3}\cos(3x) + C}$

2. (10 pts.) Suppose you have 120 feet of fencing material to enclose two rectangular regions, as illustrated. Find the dimensions x and y that maximize the total enclosed area.



Constraint: $2x + 3y = 120$

$3y = 120 - 2x$

$y = 40 - \frac{2}{3}x$

Area $= xy = x(40 - \frac{2}{3}x)$

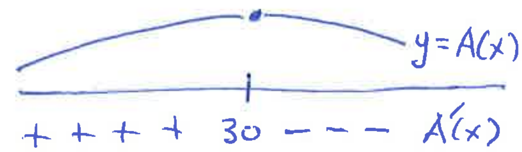
Area $= A(x) = 40x - \frac{2}{3}x^2$

Thus we need to find the x that maximizes $A(x)$ on the interval $[0, 120]$.

$A'(x) = 40 - \frac{4}{3}x = 0$

$40 = \frac{4}{3}x$

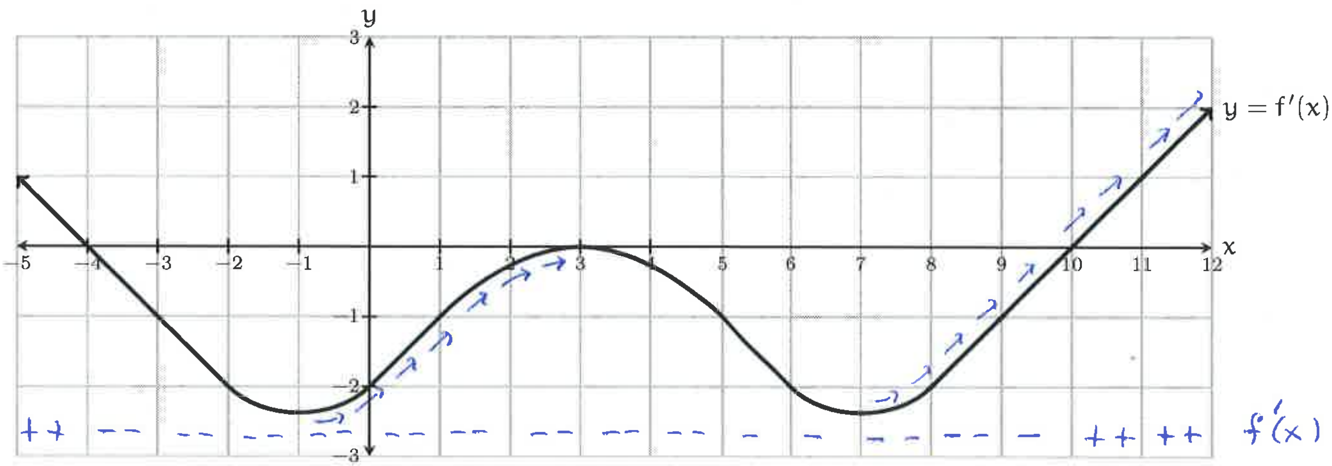
$x = 30$ ← critical point



Thus area maximized when $x = 30$, $y = 40 - \frac{2}{3} \cdot 30 = 20$

$\boxed{\text{Answer: } x = 30 \quad y = 20 \text{ for maximum enclosed area}}$

3. (10 pts.) The graph $y = f'(x)$ of **the derivative** of a function $f(x)$ is shown. Answer the questions about $f(x)$.



$f'(x)$

(a) State the intervals on which $f(x)$ increases. $\boxed{(-\infty, -4) \text{ and } (10, \infty)}$ because that's where $f'(x) > 0$

(b) State the intervals on which $f(x)$ decreases. $\boxed{(-4, 3) \text{ and } (3, 10)}$ because that's where $f'(x) < 0$

(c) List all critical points of $f(x)$. $\boxed{-4, 3, 10}$ because that's where $f'(x) = 0$.

(d) At which of these critical points is there a local maximum? $\boxed{x = -4}$ by First derivative test.

(e) State the intervals on which the function $f(x)$ is concave up. $\boxed{(-1, 3) \text{ and } (7, \infty)}$ because $f'(x)$ increases on these intervals, and therefore $f''(x) > 0$ there.

4. (20 pts.) Find the limits.

(a) $$\lim_{x\to 0} \frac{3x^2}{\cos(x)-1} = \lim_{x\to 0}\frac{6x}{-\sin(x)} = \lim_{x\to 0}\frac{6}{-\cos(x)} = \frac{6}{-\cos(0)} = \frac{6}{-1} = \boxed{-6}$$

form $\frac{0}{0}$ · Apply L'Hôpital · form $\frac{0}{0}$ · Apply L'Hôpital

(b) $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln((1+x)^{\frac{1}{x}})} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = \lim_{x\to 0} e^{\frac{\ln(1+x)}{x}}$$

form $\frac{0}{0}$

$$= \lim_{x\to 0} e^{\frac{\frac{1}{1+x}}{1}} = \lim_{x\to 0} e^{\frac{1}{1+x}} = e^{\frac{1}{1+0}} = e^{1} = \boxed{e}$$

5. (8 pts.) Is the following equation true or false?

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx = \cos\left(\frac{1}{x}\right) + C$$

Explain.

Observe that $\frac{d}{dx}\left[\cos\left(\frac{1}{x}\right)+c\right] = -\sin\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) + 0 = \frac{\sin\left(\frac{1}{x}\right)}{x^2}$, so $\boxed{\text{YES}}$, the statement is $\boxed{\text{True}}$.

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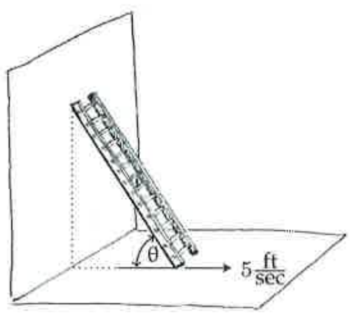
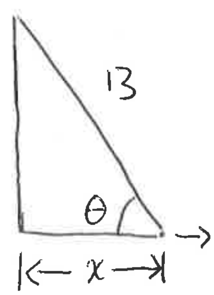
6. (10 pts.) A 13-foot ladder is leaning against a wall, as illustrated, when its base begins to slide away from the wall at a rate of 5 feet per second. At what rate is the angle θ changing when the base is 12 feet from the wall?





Know $\frac{dx}{dt} = 5$

Want $\frac{d\theta}{dt}$ when $x = 12$

From diagram, $\cos(\theta) = \frac{x}{13}$

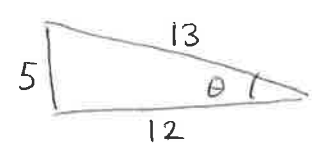
$$\frac{d}{dt}\left[\cos(\theta)\right] = \frac{d}{dt}\left[\frac{1}{13}x\right]$$

$$-\sin(\theta)\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$$

$$-\sin(\theta)\frac{d\theta}{dt} = \frac{1}{13}\cdot 5$$

$$\frac{d\theta}{dt} = \frac{-5}{13\sin(\theta)}$$

To find $\frac{d\theta}{dt}$ we just need to find $\sin(\theta)$ and plug it in above. When the base x is 12 the triangle looks like this:



Thus $\sin(\theta) = \frac{5}{13}$ $\left(= \frac{\text{OPP}}{\text{HYP}}\right)$

Therefore

$$\frac{d\theta}{dt} = \frac{-5}{13\sin\theta} = \frac{-5}{13\cdot\frac{5}{13}} = \boxed{-1\ \frac{\text{rad}}{\text{sec}}}$$