MATH 200
Calculus I

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Test 3

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Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a $\boxed{\text{box,}}$ where appropriate.

7. (10 pts.) Suppose $f(x)$ is a function for which $f'(x) = \sqrt{x} + 2$ and $f(4) = 7$. Find $f(x)$.

$f'(x) = x^{1/2} + 2$

$$f(x) = \int f'(x)\,dx = \int (x^{\frac{1}{2}} + 2)\,dx$$

$$= \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + 2x + C$$

$$= \frac{1}{3/2} x^{3/2} + 2x + C$$

$$f(x) = \frac{2}{3}\sqrt{x}^{\,3} + 2x + C$$

Now we just need to find C:

$$7 = f(4) = \frac{2}{3}\sqrt{4}^{\,3} + 2\cdot 4 + C$$

$$7 = \frac{16}{3} + 8 + C$$

$$C = -\frac{16}{3} - 1 = -\frac{16}{3} - \frac{3}{3} = -\frac{19}{3}$$

$$\boxed{f(x) = \frac{2}{3}\sqrt{x}^{\,3} + 2x - \frac{19}{3}}$$

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1. (32 points) Find the indefinite integrals.

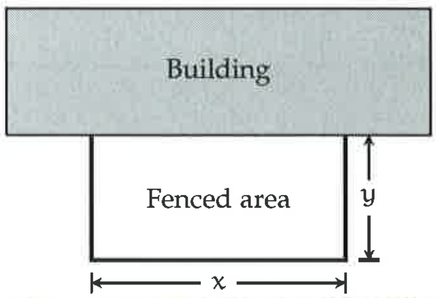
(a) $\int (x^5 + x + 1)\,dx = \frac{1}{6}x^6 + \frac{1}{2}x^2 + x + C = \boxed{\frac{x^6}{6} + \frac{x^2}{2} + x + C}$

(b) $\int 4e^{3x}\,dx = 4\int e^{3x}\,dx = 4\cdot\frac{1}{3}e^{3x} + C = \boxed{\frac{4}{3}e^{3x} + C}$

(c) $\int \frac{5}{1+x^2}\,dx = 5\int \frac{1}{1+x^2}\,dx = \boxed{5\tan^{-1}(x) + C}$

(d) $\int \left(\frac{1}{x} + \cos(x)\right)dx = \int (x^{-1} + \cos(x))\,dx = \boxed{\ln|x| + \sin(x) + C}$

2. (10 pts.) Suppose you have 160 feet of fencing material to enclose a rectangular region. One side of the rectangle will border a building, so no fencing is required for that side. Find the dimensions x and y that maximize the fenced area.



$x + 2y = 160$

$2y = 160 - x$

$y = 80 - \frac{1}{2}x$

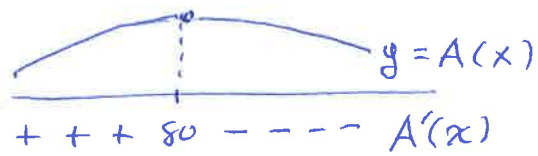
Area $= xy = x\left(80 - \frac{1}{2}x\right)$

Area $= A(x) = 80x - \frac{1}{2}x^2$

We need to maximize this on $[0, 160]$

$A'(x) = 80 - x = 0$

$x = 80 \leftarrow$ critical point



Thus maximum area when $x = 80$ and $y = 80 - \frac{1}{2}80 = 40$.

Answer $\boxed{x = 80,\ y = 40}$ maximizes area.

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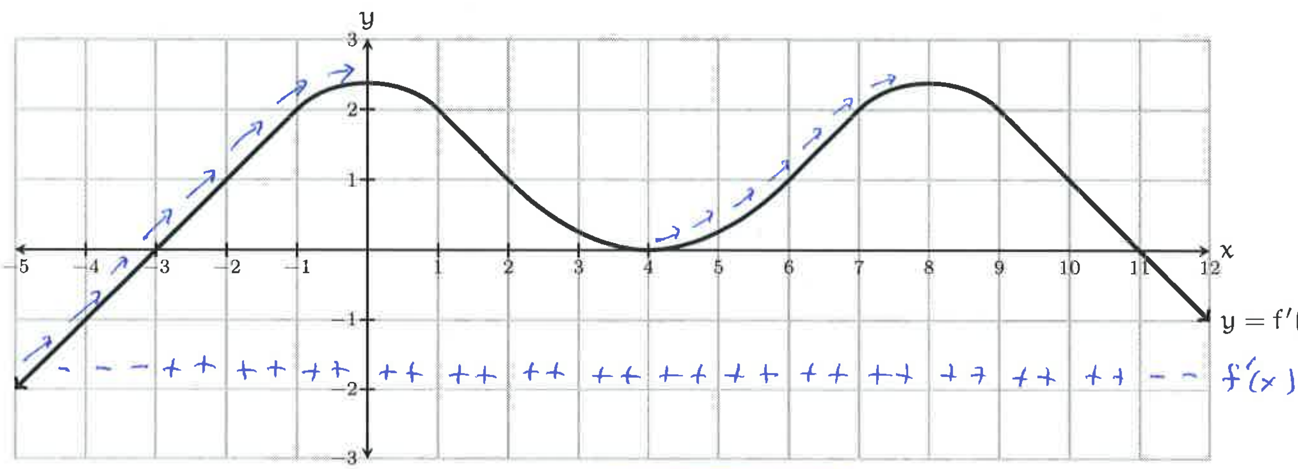
3. (10 pts.) The graph $y = f'(x)$ of **the derivative** of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) State the intervals on which $f(x)$ increases. $\boxed{(-3, 4), (4, 11)}$ because that's where $f'(x) > 0$

(b) State the intervals on which $f(x)$ decreases. $\boxed{(-\infty, -3), (11, \infty)}$ because that's where $f'(x) < 0$

(c) List all critical points of $f(x)$. $\boxed{-3, 4, 11}$ because that's where $f'(x) = 0$.

(d) At which of these critical points is there a local maximum? $\boxed{x = 11}$ by First Derivative Test.

(e) State the intervals on which the function $f(x)$ is concave up.

$f(x)$ is concave up on $\boxed{(-\infty, 0) \text{ and } (4, 8)}$ because $f'(x)$ increases on those intervals and therefore $f''(x) > 0$ there.

4. (20 pts.) Find the limits.

(a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{0+\sin(x)}{2x} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{\cos(0)}{2} = \boxed{\frac{1}{2}}$

form $\frac{0}{0}$ Apply L'Hôpital form $\frac{0}{0}$ Apply L'Hôpital

(b) $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x\ln(x)} = \lim_{x\to 0^+} e^{\frac{\ln x}{1/x}}$ form $\frac{0}{0}$

$= \lim_{x\to 0^+} e^{\frac{\frac{1}{x}}{-1/x^2}} = \lim_{x\to 0^+} e^{-x} = e^0 = \boxed{1}$

5. (8 pts.) Is the following equation true or false?

$$\int \left(\cos(x)\frac{1}{x} - \sin(x)\ln(x)\right) dx = \cos(x)\ln(x) + C$$

Explain.

Note that $\frac{d}{dx}\left[\cos(x)\ln(x)\right] =$

$= -\sin(x)\ln(x) + \cos(x)\frac{1}{x}$

$= \cos(x)\frac{1}{x} - \sin(x)\ln(x)$, so $\boxed{\text{YES}}$, it's true.

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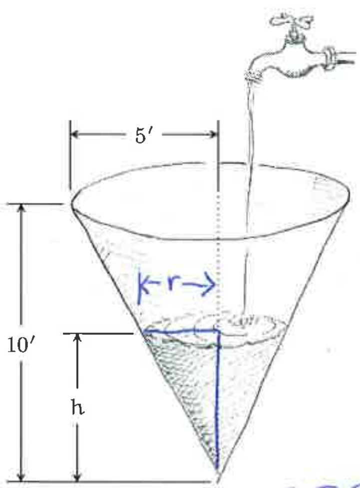
6. (10 pts.) Water flows into the conical tank (shown below) at a rate of 9 cubic feet per minute. How fast is the water level h rising when the water is 6 feet deep?



By similar triangles, $\frac{r}{h} = \frac{5}{10}$, so $\boxed{r = \frac{1}{2}h}$

The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$

Know $\frac{dV}{dt} = 9$

Want $\frac{dh}{dt}$ when $h = 6$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12}h^3$

$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{12}h^3\right]$

$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt}$

$9 = \frac{\pi}{12}3\cdot 6^2\frac{dh}{dt}$

$9 = 9\pi\frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{9}{9\pi} = \frac{1}{\pi} \text{ feet/min}}$