VCU

MATH 200

Calculus I

R. Hammack

Test 3

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Name: Richard

Score: 100

Directions. Answer the questions in the provided space. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Please put all phones away.

1. (25 points) Find the indefinite integrals.

(a) $\int \left(x^2 + \frac{1}{x^2} + e \right) dx = \int (x^2 + x^{-2} + e)\, dx$

$= \frac{x^3}{3} + \frac{x^{-2+1}}{-2+1} + ex + C = \boxed{\frac{x^3}{3} - \frac{1}{x} + ex + C}$

(b) $\int \sqrt[5]{x^3}\, dx = \int x^{\frac{3}{5}} dx = \frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1} + C$

$= \frac{x^{8/5}}{8/5} + C = \boxed{\frac{5}{8}\sqrt[5]{x}^{\,8} + C}$

(c) $\int \frac{e^{2x} + e^{-2x}}{e^x}\, dx = \int \left(\frac{e^{2x}}{e^x} + \frac{e^{-2x}}{e^x} \right) dx$

$= \int \left(e^{2x-x} + e^{-2x-x} \right) dx = \int (e^x + e^{-3x})\, dx$

$= \boxed{e^x - \frac{1}{3} e^{-3x} + C}$

(d) $\int 2 \sec^2(3x) dx = 2 \int \sec^2(3x)\, dx$

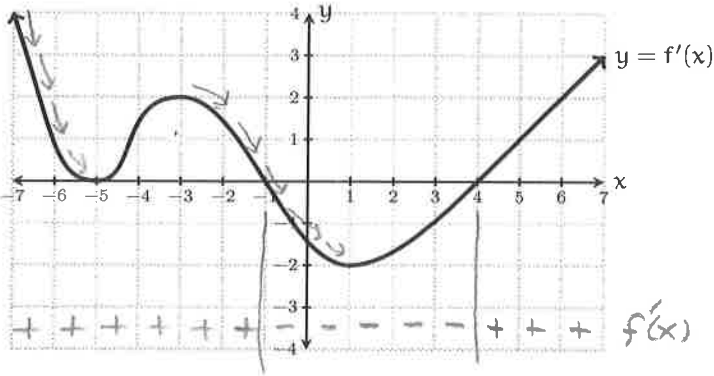
$= 2 \frac{1}{3} \tan(3x) + C = \boxed{\frac{2 \tan(3x)}{3} + C}$

(e) If $f(x)$ and $g(x)$ are differentiable functions, then

$\int f'(g(x))g'(x)\, dx = f(g(x)) + C$

chain rule in reverse

2. (15 pts.) The graph of the derivative $f'(x)$ of a function $f(x)$ is sketched below. Answer the following questions about the function $f(x)$.



(a) List the critical points of $f(x)$. $\boxed{-5, \quad -1 \quad 4}$

(Because $f'(-5)=0$, $f'(-1)=0$, $f'(4)=0$)

(b) State the interval(s) on which $f(x)$ increases.

$\boxed{(-\infty,-1) \text{ and } (4,\infty)}$

(Because $f'(x) \geq 0$ there)

(c) State the interval(s) on which $f(x)$ decreases.

$\boxed{(-1, 4)}$

(Because $f'(x) \leq 0$ there)

(d) State the locations of the relative extrema of $f(x)$.

Local max at $x = -1$
Local min at $x = 4$

(e) State the interval(s) on which $f(x)$ is concave down.

$\boxed{(-\infty, -5) \text{ and } (-3, 1)}$

(Reason: on these intervals $f'(x)$ is decreasing so its derivative $f''(x)$ is negative, i.e. $f''(x) \leq 0$, so $f(x)$ is concave down)

3. (15 pts.) The function $f(x) = x\ln(x)$ has domain $(0, \infty)$.

(a) Find the critical points of $f(x)$.

$$f'(x) = (1)\ln(x) + x\,\frac{1}{x} = \ln(x) + 1$$

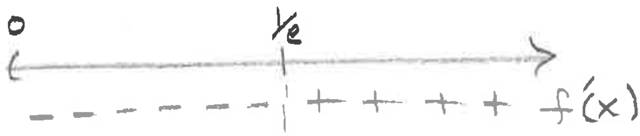
Solve: $\ln(x) + 1 = 0$

$$\ln(x) = -1$$

$$e^{\ln(x)} = e^{-1}$$

$$x = e^{-1} = \boxed{\frac{1}{e}}$$

(b) Find the intervals on which $f(x)$ increases.



Increasing on $\boxed{\left(\frac{1}{e}, \infty\right)}$

(c) Find the intervals on which $f(x)$ decreases.

Decreasing on $\boxed{\left(0, \frac{1}{e}\right)}$

(d) State the locations of the local maxima of $f(x)$ (if any).

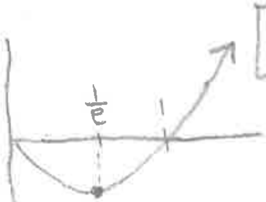
$\boxed{\text{No local max}}$

(e) State the locations of the local minima of $f(x)$ (if any).

$\boxed{\text{Local min at } x = \frac{1}{e}}$

(f) State the absolute extrema of $f(x)$ (if any).

$\boxed{\text{Absolute min at } x = \frac{1}{e}}$



4. (10 pts.) Use L'Hôpital's rule to find the limits.

(a) $\lim_{x \to \pi} \dfrac{\cos(x) + 1}{(x - \pi)^2} =$ ← form $\frac{0}{0}$

$= \lim_{x \to \pi} \dfrac{-\sin(x) + 0}{2(x - \pi)'}$ ← Apply L.R

$= \lim_{x \to \pi} \dfrac{-\sin(x)}{2x - 2\pi}$ ← form $\frac{0}{0}$

$= \lim_{x \to \pi} \dfrac{-\cos(x)}{2}$ ← Apply L.R.

$= \dfrac{-\cos(\pi)}{2} = \boxed{\dfrac{1}{2}}$

(b) $\lim_{x \to 0} \csc(7x)\sin(6x) =$ ← form $\infty \cdot 0$

$= \lim_{x \to 0} \dfrac{\sin(6x)}{\dfrac{1}{\csc(7x)}}$

$= \lim_{x \to 0} \dfrac{\sin(6x)}{\sin(7x)}$ ← form $\frac{0}{0}$

$= \lim_{x \to 0} \dfrac{\cos(6x)\,6}{\cos(7x)\,7}$ ← Apply L.R.

$= \dfrac{\cos(0)\,6}{\cos(0)\,7} = \dfrac{1 \cdot 6}{1 \cdot 7} = \boxed{\dfrac{6}{7}}$

5 (15 pts.) Find positive numbers x and y for which $xy = 12$, and the sum $2x + y$ is as small as possible.

Minimize:

$$2x + y = 2x + \frac{12}{x}$$

$y = \frac{12}{x}$

$$f(x) = 2x + \frac{12}{x}$$

Find abs. min of this on $(0, \infty)$

$$f'(x) = 2 - \frac{12}{x^2} = 0$$

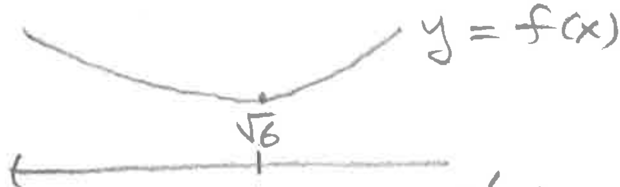
$$2 = \frac{12}{x^2}$$

$$2x^2 = 12$$

$$x^2 = 6$$

$$x = \sqrt{6}$$

critical point



$0 \quad - \; - \; - \; - \quad | \quad + \; + \; + \qquad f'(x) = 2 - \frac{12}{x^2}$

Absolute min for $x = \sqrt{6}$

$$y = \frac{12}{\sqrt{6}} = \frac{12}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{12}{6}\sqrt{6} = 2\sqrt{6}$$

$\boxed{\text{Answer: } x = \sqrt{6}, \quad y = 2\sqrt{6}}$

6. (10 pts.) A spherical balloon is inflated and its volume increases at a rate of 15 cubic inches per minute. What is the rate of change of its radius when the radius is 10 inches?

Know $\frac{dV}{dt} = 15$

Want $\frac{dr}{dt}$ when $r = 10$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$15 = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{15}{4\pi r^2}$$

$$= \frac{15}{4\pi\, 10^2} = \frac{15}{400\pi}$$

$$= \boxed{\frac{3}{80\pi} \text{ inches/min}}$$

Sphere formulas: Volume $= \frac{4}{3}\pi r^3$ Area $= \frac{1}{3}\pi r^2$

7. (10 pts.) Suppose $f(x)$ is a function for which

$$f'(x) = \frac{1}{x} + \sin(\pi x) \text{ and } f(1) = \frac{2}{\pi}. \quad \text{Find } f(x).$$

$$f(x) = \int \left(\frac{1}{x} + \sin(\pi x) \right) dx$$

$$f(x) = \ln|x| - \frac{1}{\pi}\cos(\pi x) + C$$

To find C, note that

$$\frac{2}{\pi} = f(1) = \ln|1| - \frac{1}{\pi}\cos(\pi \cdot 1) + C$$

$$\frac{2}{\pi} = 0 - \frac{1}{\pi}(-1) + C$$

$$C = \frac{2}{\pi} - \frac{1}{\pi} = \frac{1}{\pi}$$

Therefore $\boxed{f(x) = \ln|x| - \frac{1}{\pi}\cos(\pi x) + \frac{1}{\pi}}$