VCU

MATH 200

Calculus I

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Test 2

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Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Please put all phones away.

1. (20 points) Warmup: short answer.

(a) If $f(x) = e^5 + \ln(x) + x^3$, then $f'(x) = 0 + \frac{1}{x} + 3x^2 = \boxed{\frac{1}{x} + 3x^2}$

(b) If $f(x) = \ln(x)$, then $f'(3) = \boxed{\frac{1}{3}}$

Because $f'(x) = \frac{1}{x}$
So $f'(3) = \frac{1}{3}$

(c) $\lim_{h \to 0} \frac{e^{\ln(3)+h} - e^{\ln(3)}}{h} = e^{\ln(3)} = \boxed{3}$

This limit is $f'(\ln(3))$ where $f(x) = e^x$.
As $f'(x) = e^x$ we have $f'(\ln(3)) = e^{\ln 3}$
$= 3$

(d) $\frac{d}{dx}[\sin^{-1}(x)] = \boxed{\frac{1}{\sqrt{1 - x^2}}}$

(e) $\frac{d}{dx}[5^x] = \boxed{5^x \ln(x)}$

(f) $\frac{d}{dx}[\tan(x)] = \boxed{\sec^2(x)}$

(g) $\frac{d}{dx}[\sqrt[3]{x^5}] = \frac{d}{dx}\left[x^{\frac{5}{3}}\right] = \frac{5}{3} x^{\frac{5}{3} - 1} = \frac{5}{3} x^{\frac{2}{3}} = \boxed{\frac{5}{3}\sqrt[3]{x^2}}$

(h) $\frac{d}{dx}[\sec(\pi x)] = \sec(\pi x)\tan(\pi x)\,\pi$

(i) $\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -x^{-1-1} = -x^{-2} = \boxed{-\frac{1}{x^2}}$

(j) $\frac{d}{dx}[5x^3e^x] = 15x^2e^x + 5x^3e^x$

$= \boxed{5x^2e^x(3 + x)}$

$y = x^{-1}$

2. (5 points) Find the equation of the tangent line to the graph of $y = \frac{1}{x}$ at the point where $x = 2$.

$$\frac{dy}{dx} = -x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Slope where $x=2$ is $m = -\frac{1}{2^2} = -\frac{1}{4}$

Point on tangent: $(2, \frac{1}{2})$

Point-slope formula:

$$y - \frac{1}{2} = -\frac{1}{4}(x-2)$$

$$y - \frac{1}{2} = -\frac{1}{4}x + \frac{1}{2}$$

$$\boxed{y = -\frac{1}{4}x + 1}$$

3. (5 points) Information about functions $f(x)$, $g(x)$ and their derivatives is given in the table below.

x	0	1	2	3	4	5
$f(x)$	-4	-2	0	1	1	0
$f'(x)$	2	1	1	3	0.5	-1
$g(x)$	10	9	7	4	0	-4
$g'(x)$	0	-0.5	-1	-3	-4	-4

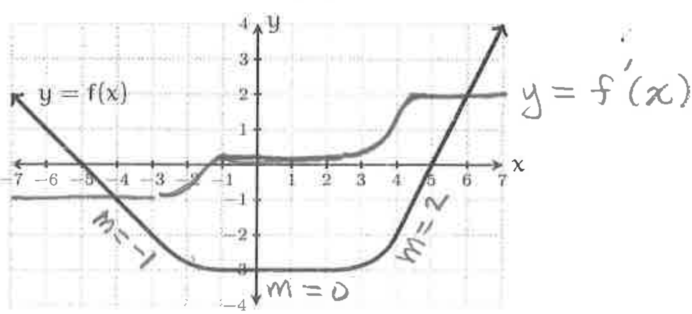
Suppose $h(x) = f(x)g(x)$. Find $h'(3)$. Show your work.

$$h'(x) = f'(x)g(x) + f(x)g'(x) \leftarrow \text{product rule}$$

$$h'(3) = f'(3)g(3) + f(3)g'(3)$$

$$= 3 \cdot 4 + 1 \cdot (-3) = \boxed{9}$$

4. (5 points) A function $f(x)$ is graphed below. Using the same coordinate axis, sketch the graph of the derivative $f'(x)$.



5. (20 points) Find the following derivatives.

(a) $\frac{d}{dx}\left[\tan^{-1}(\ln(x)) + \pi\right] = \frac{1}{1+(\ln(x))^2} \frac{d}{dx}\left[\ln x\right]$

$$= \boxed{\frac{1}{1+(\ln(x))^2} \frac{1}{x}}$$

(b) $\frac{d}{dx}\left[x^2 (\cos(x))^5\right] = 2x(\cos(x))^5 + x^2 \frac{d}{dx}\left[(\cos(x))^5\right]$

$$= \boxed{2x(\cos(x))^5 + x^2\, 5(\cos(x))^4 (-\sin(x))}$$

$$= \boxed{2x \cos^5(x) - 5x^2 \cos^4(x) \sin(x)}$$

(c) $\frac{d}{dx}\left[\frac{x^2-4x}{e^{3x}}\right] = \frac{(2x-4)e^{3x} - (x^2-4x)e^{3x} \cdot 3}{(e^{3x})^2}$

$$= \frac{e^{3x}\left((2x-4) - 3(x^2-4x)\right)}{e^{3x} e^{3x}} = \boxed{\frac{-3x^2+14x-4}{e^{3x}}}$$

(d) $\frac{d}{dx}\left[\ln(\sin^3(x))\right] = \frac{1}{\sin^3(x)} \frac{d}{dx}\left[\sin^3(x)\right]$

$$= \frac{1}{\sin^3(x)} 3\sin^2(x)(\cos(x))$$

$$= \frac{3\cos(x)}{\sin(x)} = \boxed{3\cot(x)}$$

6 (10 points) Use logarithmic differentiation to differentiate $y = \big(\sin(x)\big)^{x}$.

$$y = (\sin(x))^{x}$$

$$\ln(y) = \ln\left((\sin(x))^{x}\right)$$

$$\ln(y) = x \ln(\sin(x))$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[x \ln(\sin(x))\right]$$

$$\frac{y'}{y} = (1)\ln(\sin(x)) + x \frac{1}{\sin(x)} \cos(x)$$

$$y' = y\left(\ln(\sin(x)) + x\cot(x)\right)$$

$$\boxed{y' = (\sin(x))^{x}\left(\ln(\sin(x)) + x\cot(x)\right)}$$

7 (10 points) Recall: the derivative of $f(x)$ is $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

Use this to find derivative of the function $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

8. (15 points) An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 3\sqrt[3]{t^4} + 4t$ feet. How far away from the starting point is it when its velocity is 12 feet per second?

$$S(t) = 3t^{\frac{4}{3}} + 4t$$

$$\text{Velocity} = V(t) = S'(t) = 4t^{\frac{1}{3}} + 4$$

$$= 4\sqrt[3]{t} + 4$$

To find the time when velocity is 12 ft/sec, we solve the equation

$$V(t) = 12$$

$$4\sqrt[3]{t} + 4 = 12$$

$$4\sqrt[3]{t} = 8$$

$$\sqrt[3]{t} = 2$$

$$\left(\sqrt[3]{t}\right)^3 = 2^3$$

$t = 8$ seconds.

Thus object has velocity 12 ft/sec when $t = 8$. At this time it is at the location $S(8) = 3\sqrt[3]{8}^{\,4} + 4 \cdot 8$

$$= 3 \cdot 2^4 + 32 = 3 \cdot 16 + 32 = 48 + 32$$

$= \boxed{\text{80 feet from starting Point}}$

$y = f(x)$

9. (10 points) This question concerns the equation $\cos(y) + x^2 + x = e^y$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[\cos(y) + x^2 + x\right] = \frac{d}{dx}\left[e^y\right]$$

$$-\sin(y)\frac{dy}{dx} + 2x + 1 = e^y \frac{dy}{dx}$$

$$2x + 1 = \sin(y)\frac{dy}{dx} + e^y \frac{dy}{dx}$$

$$2x + 1 = \left(\sin(y) + e^y\right)\frac{dy}{dx}$$

$$\boxed{\frac{2x+1}{\sin(y)+e^y} = \frac{dy}{dx}}$$

(b) Use your answer from part (a) to find the slope of the tangent line to the graph of $\cos(y) + x^2 + x = e^y$ at the point $(0, 0)$.

$$\left.\frac{dy}{dx}\right|_{(0\,0)} = \frac{2 \cdot 0 + 1}{\sin(0) + e^0} = \frac{1}{0+1} = \boxed{1}$$