VCU

MATH 200

CALCULUS I

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TEST 2

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Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. Put your final answer in a $\boxed{\text{box}}$ when appropriate.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Please put all phones away.

1. (20 points) Warmup: short answer.

(a) If $f(x) = x^3 + \ln(x) + \pi^3$, then $f'(x) = \boxed{3x^2 + \frac{1}{x}}$

(b) If $f(x) = e^x$, then $f'(\ln(3)) = e^{\ln 3} = \boxed{3}$

(c) $\displaystyle\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+h\right) - \sin\left(\frac{\pi}{3}\right)}{h} = \cos\left(\frac{\pi}{3}\right) = \boxed{\frac{1}{2}}$

The limit is $f'\left(\frac{\pi}{3}\right)$, where $f(x) = \sin(x)$. Thus $f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

(d) $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \boxed{\frac{1}{|x|\sqrt{x^2-1}}}$

(e) $\frac{d}{dx}\left[3^x\right] = \boxed{3^x \ln(3)}$

(f) $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \boxed{\frac{1}{1+x^2}}$

(g) $\frac{d}{dx}\left[\sqrt[3]{x^5}\right] = \frac{d}{dx}\left[x^{\frac{5}{3}}\right] = \frac{5}{3}x^{\frac{5}{3}-1} = \frac{5}{3}x^{\frac{2}{3}} = \boxed{\frac{5}{3}\sqrt[3]{x^2}}$

(h) $\frac{d}{dx}\left[\cos(\pi x)\right] = \boxed{-\sin(\pi x)\pi}$

(i) $\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -x^{-1-1} = -x^{-2} = \boxed{-\frac{1}{x^2}}$

(j) $\frac{d}{dx}\left[5x^2\ln(x)\right] = 10x\ln(x) + 5x^2\frac{1}{x}$

$= \boxed{10x\ln(x) + 5x}$

2. (5 points) Find the equation of the tangent line to the graph of $y = \sqrt{x}$ at the point where $x = 9$.

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Thus slope of tangent where $x = 9$ is $m = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}$. Also, the point $(9, \sqrt{9}) = (9, 3)$ is on The tangent line. By point-slope form of a line,

$$y - 3 = \frac{1}{6}(x - 9)$$

$$y = \frac{1}{6}x - \frac{3}{2} + 3$$

$$\boxed{y = \frac{1}{6}x + \frac{3}{2}}$$

3. (5 points) Information about functions $f(x)$, $g(x)$ and their derivatives is given in the table below.

x	0	1	2	3	4	5
$f(x)$	−4	−2	0	1	1	0
$f'(x)$	2	1	1	3	0.5	−1
$g(x)$	10	9	7	4	0	−4
$g'(x)$	0	−0.5	−1	−3	−4	−4

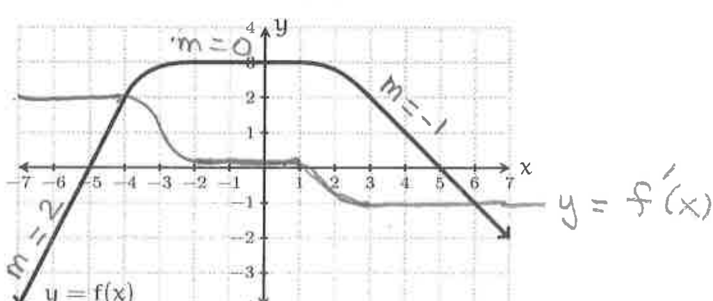
Suppose $h(x) = f(g(x))$. Find $h'(3)$. Show your work.

$h'(x) = f'(g(x))\,g'(x) \leftarrow$ chain rule

$h'(3) = f'(g(3))\,g'(3)$

$= f'(4)\,g'(3) = (0.5)(-3) = \boxed{\frac{-3}{2}}$

4. (5 points) A function $f(x)$ is graphed below. Using the same coordinate axis, sketch the graph of the derivative $f'(x)$.



5. (20 points) Find the following derivatives.

(a) $\frac{d}{dx}[\tan(\ln(x)) + x] = \boxed{\sec^2(\ln(x))\frac{1}{x} + 1}$

$$= \boxed{\frac{\sec^2(\ln(x))}{x} + 1}$$

(b) $\frac{d}{dx}\left[(x^2\sin(x))^5\right] = 5(x^2\sin(x))^4 \frac{d}{dx}\left[x^2\sin(x)\right]$

$$= \boxed{5(x^2\sin(x))^4\left(2x\sin(x) + x^2\cos(x)\right)}$$

(c) $\frac{d}{dx}\left[\frac{e^{3x}}{x^2-4x}\right] = \boxed{\frac{3e^{3x}(x^2-4x) - e^{3x}(2x-4)}{(x^2-4x)^2}}$

(d) $\frac{d}{dx}[\ln(\sin(x^3))] = \frac{\frac{d}{dx}\left[\sin(x^3)\right]}{\sin(x^3)}$

$$= \frac{\cos(x^3)\,3x^2}{\sin(x^3)}$$

$$= \boxed{\cot(x^3)\,3x^2}$$

6 (10 points) Use logarithmic differentiation to differentiate $y = (x^3 + x)^x$.

$$y = (x^3 + x)^x$$

$$\ln(y) = \ln((x^3+x)^x) = x \ln(x^3+x)$$

$$\frac{d}{dx}[y] = \frac{d}{dx}\left[x \ln(x^3+x)\right]$$

$$\frac{y'}{y} = (1)\ln(x^3+x) + x \frac{3x^2+1}{x^3+x}$$

$$y' = y\left(\ln(x^3+x) + \frac{3x^2+1}{x^2+1}\right)$$

$$y' = (x^3+x)^x\left(\ln(x^3+x) + \frac{3x^2+1}{x^2+1}\right)$$

7 (10 points) Recall: the derivative of $f(x)$ is $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

Use this to find derivative of the function $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \frac{(x+h)x}{(x+h)x}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = \boxed{\frac{-1}{x^2}}$$

8. (15 points) An object moves on a straight line in such a way that its distance from its starting point at time t seconds is $s(t) = 3\sqrt[3]{t}^4 + 4t$ feet. How far away from the starting point is it when its velocity is 12 feet per second?

$$S(t) = 3t^{\frac{4}{3}} + 4t$$

$$\text{Velocity} = V(t) = S'(t) = 4t^{\frac{1}{3}} + 4$$

$$= 4\sqrt[3]{t} + 4$$

To find the time when velocity is 12 ft/sec, we solve the equation

$$V(t) = 12$$

$$4\sqrt[3]{t} + 4 = 12$$

$$4\sqrt[3]{t} = 8$$

$$\sqrt[3]{t} = 2$$

$$(\sqrt[3]{t})^3 = 2^3$$

$$t = 8 \text{ seconds.}$$

Thus object has velocity 12 ft/sec when $t = 8$. At this time it is at the location $S(8) = 3\sqrt[3]{8}^4 + 4 \cdot 8$

$= 3 \cdot 2^4 + 32 = 3 \cdot 16 + 32 = 48 + 32$

$= \boxed{\text{80 feet from starting point}}$

9. (10 points) This question concerns the equation $\cos(y^2) + x = e^y$. $y = f(x)$

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left[\cos(y^2) + x\right] = \frac{d}{dx}\left[e^y\right]$$

$$-\sin(y^2)2y\frac{dy}{dx} + 1 = e^y\frac{dy}{dx}$$

$$1 = \sin(y^2)2y\frac{dy}{dx} + e^y\frac{dy}{dx}$$

$$1 = \left(\sin(y^2)2y + e^y\right)\frac{dy}{dx}$$

$$\boxed{\frac{1}{\sin(y^2)2y + e^y} = \frac{dy}{dx}}$$

(b) Use your answer from part (a) to find the slope of the tangent line to the graph of $\cos(y^2) + x = e^y$ at the point $(0, 0)$.

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{1}{\sin(0^2)2\cdot 0 + e^0} = \frac{1}{0+1}$$

$$= \boxed{1}$$