MATH 200
Chapter 5 Summary and Review Sheet

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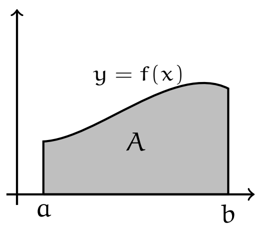
THIS guide summarizes the main topics of Chapter 5 that you should know for the final exam. But merely remembering these ideas is not sufficient preparation – you must internalize them. This is only possible if you work lots of exercises for practice. See the Exercise list on the MATH 200 web page.

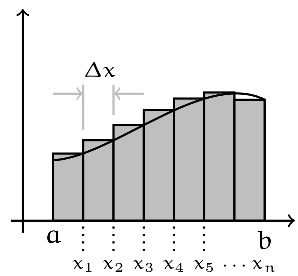
Area and The Definite Integral (Sections 5.1, 5.2)

The main topic of Chapter 5 is the **definite integral**. This idea is motivated by the problem of finding the area under the graph of a function $y = f(x)$. Here is one way to approach this problem:

Problem: Find the area A under the graph of $y = f(x)$, between $x = a$ and $x = b$.



Solution: Approximate A by the sum of the areas of n rectangles of width $\Delta x = \frac{b-a}{n}$, as illustrated. The sum of these areas is $\sum_{k=1}^{n} f(x_k)\Delta x$.



To get the exact area A, let the number n of rectangles approach ∞:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x.$$

This leads to a fundamental definition:

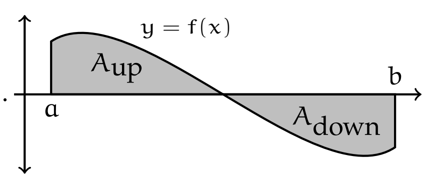
Definition. The **definite integral of** $f(x)$ **over** $[a, b]$ **is the number**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x. \qquad (*)$$

Thus $\int_a^b f(x)dx$ equals the area under $f(x)$ between a and b when $f(x) \geqslant 0$ on $[a, b]$. This number may be difficult to compute as a limit, but the Fundamental Theorem of Calculus (next page) gives a simple means of computing it without a limit. (But its limit definition gives it a meaning – namely area.)

Properties of the Definite Integral

(0) $\int_a^b f(x)dx = A_{\text{up}} - A_{\text{down}}$ ....



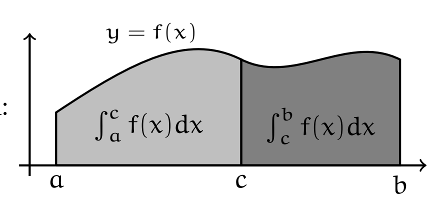
(1) $\int_a^a f(x)dx = 0$ .... (Because the area between a and a is zero.)

(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ .... (Reason: On left $\Delta x = \frac{b-a}{n}$; on right it is $\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$.)

(3) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ .... (In (*), a constant c in front of $f(x)$ will factor out of both the Σ and the limit.)

(4) $\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ .... Reason:



The Fundamental Theorem of Calculus (Section 5.4)

This theorem links several major ideas in MATH 200: derivatives, antiderivatives, definite integrals and area.
It has two parts, Part I and Part II:

Fundamental Theorem of Calculus I:
If $f(x)$ is continuous on the interval $[a, b]$, then $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

Part I (above) is not nearly as useful to us as its companion, Part II. It gives a formula for computing definite integrals:

Fundamental Theorem of Calculus II:
If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is *any* antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$.

You can interpret Fundamental Theorem of Calculus II as saying

$$\int_a^b F'(x)\,dx = F(b) - F(a),$$

$$\text{that is,}\quad \int_a^b \begin{pmatrix}\text{Rate of}\\ \text{change of}\\ \text{quantity}\\ F(x)\end{pmatrix} dx = \begin{pmatrix}\text{Net change of}\\ \text{quantity } F(x)\\ \text{between } x = a\\ \text{and } x = b\end{pmatrix}.$$

For example, if $s(t)$ gives position of a moving object at time t, then the velocity at time t is $v(t) = s'(t)$, and $\int_a^b v(t)\,dt = s(b) - s(a)$ is the distance between the object's location at time a and its location at time b. Said differently, the area under $v(t)$ between a and b is the distance between the object's locations at times a and b.

Average Value of a Function. You can also use a definite integral to compute the average value of a function:
The average value of $f(x)$ on the interval $[a, b]$ is $\dfrac{1}{b-a}\int_a^b f(x)\,dx$.

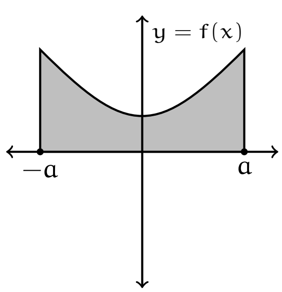
The Substitution Method (Sections 5.5)

Substitution in indefinite integrals (the chain rule in reverse).
If $u = g(x)$, then $\int f\big(g(x)\big) \cdot g'(x)\,dx = \int f(u)\,du$.
(This is important; get lots of practice using it! See examples from class; work the exercises.)

Substitution in definite integrals: $\int_a^b f\big(g(x)\big) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Even and Odd Functions (Section 5.4)

A function $f(x)$ is **even** if $f(-x) = f(x)$ for all x in its domain. If $f(x)$ is a even function, continuous on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.



A function $f(x)$ is **odd** if $f(-x) = -f(x)$ for all x in its domain. If $f(x)$ is an odd function, continuous on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 0$.