# MATH 200
# Chapter 4 Summary and Review

R. Hammack
A. Hoeft

THIS guide summarizes the topics of Chapter 4 that may be represented on the test. The test covers Section 3.11 and all sections of Chapter 4, with the exception of 4.5 and 4.8. In studying, please remember that merely remembering these facts and ideas is not sufficient preparation for the test – you must internalize them and apply them. This is only possible if you work lots of exercises for practice. See the Exercise list on the MATH 200 web page.

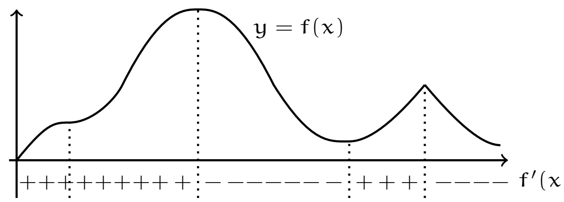
## How a Function's Derivative Determines its Shape

Sections 4.1, 4.2, 4.3 and 4.4 are concerned with one main theme: Obtain information about a function $f(x)$ by examining its derivatives $f'(x)$ and $f''(x)$. Typically, we begin with a question about $f(x)$, then use its derivatives to obtain an answer. Once we get our final answer we can forget about $f'(x)$ and $f''(x)$ because they were just part of the process of finding an answer, not a part of the answer itself. (But you are *still* expected to show your work!)

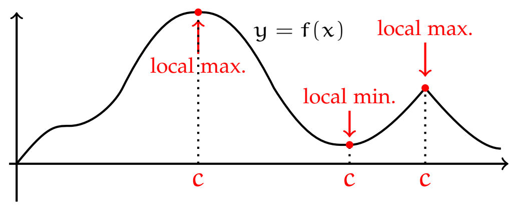
1. **Increasing/Decreasing**

   Because $f'(x)$ equals the slope of the tangent to the graph of $y = f(x)$ at $x$, we get the following interpretation:

   - Wherever $f'(x) > 0$, the function $f(x)$ is increasing.
   - Wherever $f'(x) < 0$, the function $f(x)$ is decreasing.




2. **Local Extrema**
   - If $(c, f(c))$ is higher than nearby points on $y = f(x)$, we say $f(c)$ is a **local maximum** of $f(x)$ occurring at $x = c$. Any local maximum is "at the top of a hill."
   - If $(c, f(c))$ is lower than nearby points on $y = f(x)$, we say $f(c)$ is a **local minimum** of $f(x)$ occurring at $x = c$. Any local minimum is "at the bottom of a valley."

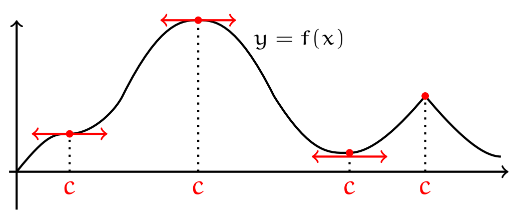



3. **Critical Points**

   A **critical point** of $f(x)$ is a number $c$ in the domain of $f(x)$ for which $f'(c)$ is either zero or undefined.

   The four critical points $c$ of a function are indicated on the right.

   *Although not all critical points are the locations of local extrema, all local extrema occur at critical points.*

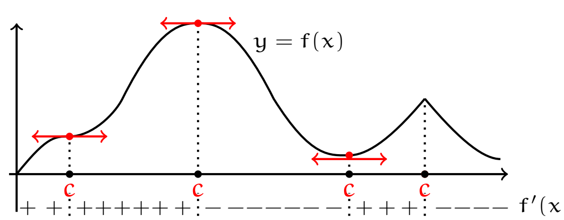



   To find the critical points of $f(x)$, first compute $f'(x)$. Examine it and identify all numbers $x = c$ (if any) for which $f'(c)$ is not defined. If this $c$ is in the domain of the original $f(x)$, add this $c$ to your growing list of critical points. Then solve the equation $f'(x) = 0$. Each solution is a critical point $x = c$ that makes $f'(c) = 0$ and add these to your list of critical points, too. This list now contains all candidate locaitons for local extrema of $f(x)$.

4. **The First Derivative Test**

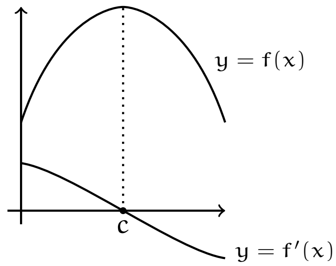
   Combining the above information and pictures, we get:

   | |
   |---|
   | **The First Derivative Test** (For finding all local extrema.) |
   | Suppose $c$ is a critical point of $f(x)$. |
   | 1. If $f'(x)$ goes from $+$ to $-$ at $c$, then $f(x)$ has a **local max.** of $f(c)$ at $c$.<br>2. If $f'(x)$ goes from $-$ to $+$ at $c$, then $f(x)$ has a **local min.** of $f(c)$ at $c$.<br>3. If $f'(x)$ does not change sign at $c$, there is no local extremum at $c$. |




   Therefore, to find all local extrema of a function $f(x)$, first find its critical points $c$. Then examine how the sign of $f'(x)$ changes around each $c$, and draw a conclusion based on the First Derivative Test. Bear in mind that there is no guarantee that $f'(x)$ will change sign at $c$. (See the left-most $c$ in the above drawing.) If there is no sign change at $c$, the First Derivative Test guarantees that there is no local extremum at $c$.

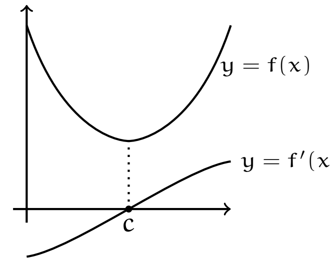
Where $f(x)$ is concave down, its slopes decrease. (On the right, slope changes from positive to negative.) Thus the function $f'(x)$ decreases. Thus its derivative $f''(x)$ is negative.




**Conclusion:**

Where $f''(x) < 0$, the function $f(x)$ is concave down.

Where $f(x)$ is concave up, its slopes increase. (On the right, slope changes from negative to positive.) Thus the function $f'(x)$ (giving slope) increases. Thus its derivative $f''(x)$ is positive.




**Conclusion:**

Where $f''(x) > 0$, the function $f(x)$ is concave up.

Notice that in the pictures above, $f(x)$ has a local maximum at $c$ where $f(x)$ is concave down (i.e., where $f''(x) < 0$). Likewise, $f(x)$ has a local minimum at $c$ where $f(x)$ is concave up (i.e., where $f''(x) > 0$). This leads to:

**The Second Derivative Test** (For finding local extrema)
Suppose $c$ is a critical point of $f(x)$ for which $f'(x) = 0$ (as in the pictures above). Then:

1. If $f''(c) < 0$, then $f(x)$ has a local maximum at $c$.
2. If $f''(c) > 0$, then $f(x)$ has a local minimum at $c$.
3. If $f''(c) = 0$, then there is no conclusion. (There could be a local maximum, a local minimum, or neither. In such a situation you will have to use the First Derivative Test.)

The Second Derivative tests has two drawbacks. It does not apply to critical points $c$ for which $f'(c)$ is undefined, and it can be inconclusive (though this is rare). If the Second Derivative Test is inconclusive for one of your critical points, then apply the First Derivative Test to that critical point. On our exam, you can use either the first or second derivative test. Since you can always just get by with the First Derivative Test, the Second Derivative Test is optional.

## ABSOLUTE EXTREMA (SECTIONS 4.1 AND 4.4)

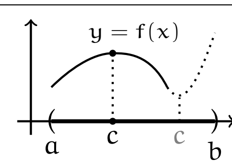
In applied optimization problems, you will need to find an absolute maximum or absolute minimum of a function.

**Definitions:** Suppose $f(x)$ is defined on a domain D (usually an interval, or perhaps a union of intervals). Then:

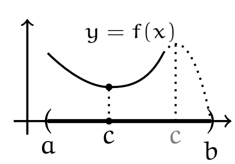
1. $f(x)$ has an **absolute maximum** at a number $c$ in D if $f(c) \geqslant f(x)$ for all $x$ in D.
2. $f(x)$ has an **absolute minimum** at a number $c$ in D if $f(c) \leqslant f(x)$ for all $x$ in Dn.

Finding the absolute extrema (and their locations) can potentially be problematic. *Recall that the absolute max is a local max which is higher than any other local max and an absolute min is a local min which is lower than any other local min.* Then in the worse case scenario, you may have to sketch the graph of $f(x)$, find the local extrema and draw a conclusion based on this information. (In general, one or both absolute extrema may not even exist.) There is one fairly common situation in which there is an easy answer. That is when you need to find the absolute extrema of $f(x)$ on an interval (open or closed), and $f(x)$ has just one critical point on the interval.

Suppose $f(x)$ has just one critical point $c$ on an interval. If $f(x)$ has a local maximum at $c$, then it is an absolute maximum. (This is so because $f(x)$ can't rise higher than $f(c)$, as indicated by the dotted part of the graph on the right, without yielding another critical point.)




Suppose $f(x)$ has just one critical pointc on an interval. If $f(x)$ has a local minimum at $c$, then it is an absolute minimum. (This is so because $f(x)$ can't sink lower than $f(c)$, as indicated by the dotted part of the graph on the right, without yielding another critical point.)

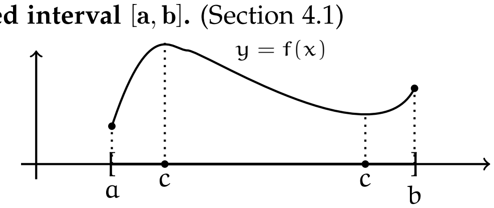



Thus, if you find that there is just one critical point, you can decide whether it yields an absolute maximum or minimum by applying the First Derivative Test (or the Second Derivative Test).

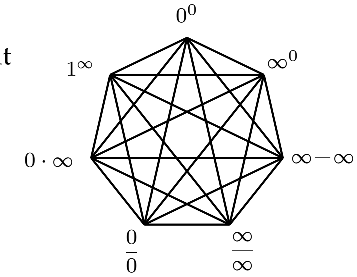
If you seek the absolute extrema of a function $f(x)$ on a **closed** interval, there is a simple method for doing this:

**How to find the absolute extrema of a function $f(x)$ defined on a closed interval $[a, b]$.** (Section 4.1)

1. Find all critical points $c$ in $[a, b]$.
2. Compute $f(c)$ for $c$ from Step 1, and also $f(a)$ and $f(b)$.
3. The largest value in Step 2 is the absolute maximum. The smallest is the absolute minimum.



## Indeterminate Forms (Section 4.7)




There are seven indeterminate forms, as shown on the right. L'Hôpital's Rule gives a way of computing limits that have such forms. Be careful not to use it on a limit that does not have an indeterminate form. L'Hôpital's Rule says that if a limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

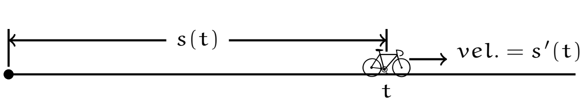
Be sure that you know how to use L'Hôpital's Rule to find the limit of an indeterminate form of any type by using algebra to shape it into one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Also, log properties can be helpful.

## Antiderivatives (Section 4.9)

The **antiderivative** of a function $f(x)$ is a function $F(x)$ whose derivative is $f(x)$.
If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ is also an antiderivative of $f(x)$ for any constant $C$.
The set of all antiderivatives of $f(x)$ is denoted $\int f(x)\,dx$. This is called the **indefinite integral** of $f(x)$.
Thus $\int f(x)\,dx = F(x) + C$ means that $\frac{d}{dx}[F(x) + C] = f(x)$.

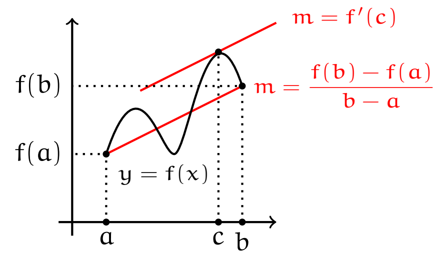
| **Derivative Rule** | **Antiderivative Rule** | |
|---|---|---|
| $\frac{d}{dx}[\,x^n\,] = nx^{n-1}$ | $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ | (for $n \neq -1$) |
| $\frac{d}{dx}[\,\ln(x)\,] = \frac{1}{x}$ | $\int x^{-1} dx = \ln\lvert x\rvert + C$ | (for $n = -1$) |
| $\frac{d}{dx}[\,e^x\,] = e^x$ | $\int e^x dx = e^x + C$ | |
| $\frac{d}{dx}[\sin(x)] = \cos(x)$ | $\int \cos x\,dx = \sin(x) + C$ | Trig Rules |
| $\frac{d}{dx}[\cos(x)] = -\sin(x)$ | $\int \sin x\,dx = -\cos(x) + C$ | |
| $\frac{d}{dx}[\tan(x)] = \sec^2(x)$ | $\int \sec^2(x)\,dx = \tan(x) + C$ | |
| $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$ | $\int \csc^2(x)\,dx = -\cot(x) + C$ | |
| $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ | $\int \sec(x)\tan(x)\,dx = \sec(x) + C$ | |
| $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$ | $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$ | |
| $\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$ | $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$ | Inverse Trig Rules (One reason inverse trig functions are significant is that they give us these antiderivative formulas.) |
| $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$ | $\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$ | |
| $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ | $\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}dx = \sec^{-1}(x) + C$ | |

**Motion on a Straight Line:** If a moving object has position $s(t)$ at time $t$, then:




$$\begin{cases} \text{position} & = s(t) = \cdots \quad \cdots \int v(t)dt \\ \text{velocity} & = v(t) = s'(t) = \int a(t)dt \\ \text{acceleration} & = a(t) = v'(t) = \int j(t)dt \\ \text{jerk} & = j(t) = a'(t) \end{cases}$$

The Mean Value Theorem states that if $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there is at least one number $c$ in $(a, b)$ for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.




This is an important result that we will use in Chapter 5. However, you should expect it to play less of a role on the test.

For now, the main application of the Mean Value Theorem is the fact (proved in the text) that if $F'(x) = G'(x)$, then $G(x) = F(x) + C$ for some constant $C$. This means that any two antiderivatives $F(x)$ and $G(x)$ of a function $f(x)$ differ by a constant.

# Overview

Be sure you have a command of the following topics.

- As background knowledge, be able to apply the various derivative rules from Chapter 3.
  Understand the slope interpretation of the derivative.
  Work competently with $e^x$ and $\ln x$, as well as the trigonometric functions and their inverses.
- Solve related rates problems (Section 3.11).
- Find the intervals on which a function increases and decreases (Section 4.2).
- Find the intervals on which a function is concave up or down (Section 4.2).
- Locate and compute local extrema (Sections 4.2).
- Sketch the graph of a function with correct concavity, etc. (Section 4.3).
- Find absolute extrema of a function on an interval (Sections 4.1, 4.4).
- Solve applied optimization problems (Section 4.4).
- Evaluate limits with indeterminate forms (Section 4.7).
- Find antiderivatives of functions (Section 4.9).

# Important Note

This review sheet stresses *facts*. You will need to be able to *apply* the facts to solve problems.