MATH 200
Chapter 2 Summary and Review Sheet

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THIS guide summarizes the main topics of Chapter 2 that you should know for the test and final exam. But be aware that merely remembering these ideas is not sufficient preparation – you must internalize them. This is only possible if you work lots of exercises for practice. See the Exercise list on the MATH 200 web page.

Limits and Limit Laws

The idea of a limit is central to calculus. It is important because the formula for the slope of a tangent line involves a limit. Consequently, the definition of a derivative, the most central idea of calculus, is phrased in terms of a limit. Limits are also used to detect and determine asymptotes of functions.

Informal Definition: $\lim_{x\to c} f(x)$ stands for the number that $f(x)$ approaches as x approaches c (if such a number exists).

Right Hand Limit: $\lim_{x\to c^+} f(x)$ stands for the number that $f(x)$ approaches as x approaches c *from the right*.

Left Hand Limit: $\lim_{x\to c^-} f(x)$ stands for the number that $f(x)$ approaches as x approaches c *from the left*.

Limit Laws. (All laws in this box also work for right and left hand limits.)
Here are two formulas that should be obvious.

(a) $\lim_{x\to c} x = c$ **(b)** $\lim_{x\to c} k = k$ (Where k is a constant.)

More complex limits can be reduced to simpler limits with the following laws.
Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. Then:

1. $\lim_{x\to c} \big(f(x)+g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c} \big(f(x)-g(x)\big) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
4. $\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$
5. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ (Provided $\lim_{x\to c} g(x) \neq 0$.)
6. $\lim_{x\to c} \big(f(x)\big)^n = \left(\lim_{x\to c} f(x)\right)^n$ (For an integer $n > 0$.)
7. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ (For an integer $n > 0$, and $\lim_{x\to c} f(x) > 0$ if n is even.)

In some cases you can evaluate (i.e., compute) a limit by using a combination of the above limit laws. For example, consider evaluating $\lim_{x\to 3}(x^2+5x)$. Using facts and laws from the above list allows us to work this out as

$$\lim_{x\to 3}(x^2+5x) \;=\; \lim_{x\to 3} x^2 + \lim_{x\to 3} 5x \;=\; (\lim_{x\to 3} x)^2 + 5\lim_{x\to 3} x \;=\; 3^2 + 5\cdot 3 = 24.$$

Notice that the answer 24 is just the value we get when we plug 3 (the number x approaches) into the function x^2+5x. In fact, similar reasoning shows that if $f(x)$ is any polynomial, then $\lim_{x\to c} f(x) = f(c)$, so limits of polynomials are very easy.

In other cases, especially in evaluating a limit of a quotient of functions, further algebraic simplifications may be necessary. Here is a summary:

- If in $\lim_{x\to c} \dfrac{f(x)}{g(x)}$, we have $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$. (Just work out the top and bottom limits.)
- If in $\lim_{x\to c} \dfrac{f(x)}{g(x)}$, we have $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then the limit $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ **does not exist**.

  (But it *could* equal ∞ or $-\infty$, meaning there is a vertical asymptote at $x = c$. More on that on page 4 of this sheet.)
- If in $\lim_{x\to c} \dfrac{f(x)}{g(x)}$, both $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then the limit $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ may or may not exist.

  To find out, try to algebraically cancel the terms in $\frac{f(x)}{g(x)}$ that make $g(x)$ approach zero. Then work out the limit using whichever limit laws apply. (Most significant limits in calculus are of this type.)

Trigonometric Limits

- $\lim_{x\to c} \sin(x) = \sin(c)$
- $\lim_{x\to c} \cos(x) = \cos(c)$

Note: These equations imply that $\sin(x)$ and $\cos(x)$ are continuous at any c.

- $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$
- $\lim_{x\to 0} \dfrac{\cos(x) - 1}{x} = 0$

Note that these can also be worked out using L'Hôpital's Rule, from Chapter 4.

Continuity

Definition: A function $f(x)$ is **continuous** at the point $x = c$ if $\lim_{x\to c} f(x) = f(c)$.
This means that the following three conditions must be met: 1. $f(c)$ is defined; 2. $\lim_{x\to c} f(x)$ exists; 3. $\lim_{x\to c} f(x) = f(c)$.

Intuitively, $f(x)$ being continuous at $x = c$ means that the graph of $f(x)$ does not have a "break" at $x = c$ and you can sketch the graph of $f(x)$ near $x = c$ without lifting your pencil.

Definitions: A function $f(x)$ is continuous on an open interval (a, b) if it's continuous at each number c in this interval.

A function $f(x)$ is continuous on a closed interval $[a, b]$ if it's continuous on (a, b), and $\lim_{x\to a^+} f(x) = f(a)$, and $\lim_{x\to b^-} f(x) = f(b)$.

Examples: Because $\lim_{x\to c} \sin(x) = \sin(c)$ for *any* number c, it follows that the function $\sin(x)$ is continuous at any number c.
Thus $\sin(x)$ is continuous on the interval $(-\infty, \infty)$. The same is true for $\cos(x)$, as well as any polynomial.

Building Continuous Functions from Continuous Functions:
If $f(x)$ and $g(x)$ are continuous on their domains, then any function constructed in any combination of the following ways is also continuous on its domain:

- $f(x) + g(x)$
- $f(x) - g(x)$
- $k \cdot f(x)$
- $f(x)g(x)$
- $\dfrac{f(x)}{g(x)}$
- $|f(x)|$
- $(f(x))^n$ (for a positive integer n)
- $\sqrt[n]{f(x)}$ (for a positive integer n)
- $f \circ g$

The main point of this theorem is that if a function is built up out by combining continuous functions using the above operations, then it itself is continuous. As an example, consider the functions $y = x$ and $y = \sin(x)$ and $y = \cos(x)$, which are continuous on their domains and thus can be used to build other continuous functions in the ways listed above. Then

$$h(x) = \frac{\cos(x) + x^2}{\sin(x)} + 5\sqrt{x}$$

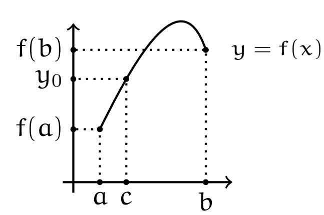
is continuous on its domain because it's built up by combining x, $\sin(x)$ and $\cos(x)$ with operations listed above.

Limit of a Composition: If $f(x)$ is continuous at $L = \lim_{x\to c} g(x)$, then $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$.

Example: Consider the problem of evaluating $\lim_{x\to 3} \cos(x^2 - 9)$.
Because $\cos(x)$ is continuous at any number, the above gives $\lim_{x\to 3} \cos(x^2 - 9) = \cos\left(\lim_{x\to 3}(x^2 - 9)\right) = \cos(0) = 1$.

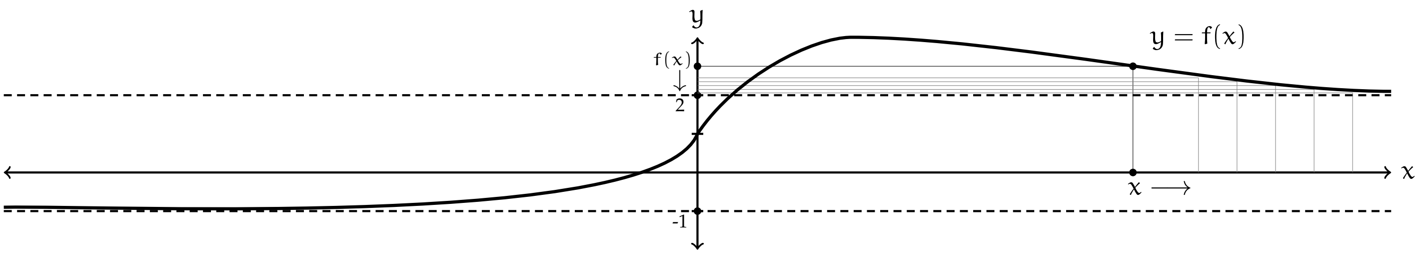
Intermediate Value Theorem: If $f(x)$ is continuous on a closed interval $[a, b]$, and y_0 is any number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ for which $f(c) = y_0$.

In particular, this means that if one of $f(a)$ and $f(b)$ is positive and the other is negative, then the equation $f(x) = 0$ has a solution c in $[a, b]$.
Reason: In such a case 0 is between $f(a)$ and $f(b)$, so the theorem says there is a number c in $[a, b]$ with $f(c) = 0$.



Limits at Infinity and Horizontal Asymptotes

For some functions $f(x)$, limits such as $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ make sense and evaluate to finite numbers. Consider the function $f(x)$ graphed below. In the long run, as x moves to the right (towards positive infinity) the corresponding $f(x)$ value approaches 2. We express this in symbols as $\lim_{x\to\infty} f(x) = 2$. Such a limit is called a *limit at infinity*, which is a bit of a misnomer because x is never "at" infinity, just approaching it. In the below picture, the graph gets closer and closer to the dashed horizontal line $y = 2$ as x moves towards ∞. This line is called a **horizontal asymptote** of the function $f(x)$. It is **not** a part of the graph, but it helps us visualize the behavior of $f(x)$ as x gets bigger and bigger.



Also, in this picture, as x moves to the left (towards negative infinity), the corresponding value $f(x)$ approaches -1. We express this in symbols as $\lim_{x\to-\infty} f(x) = -1$. As x approaches $-\infty$, the graph of $f(x)$ becomes ever closer to the horizontal line $y = -1$, which is a second horizontal asymptote of this function $f(x)$.

Summary of limits at infinity

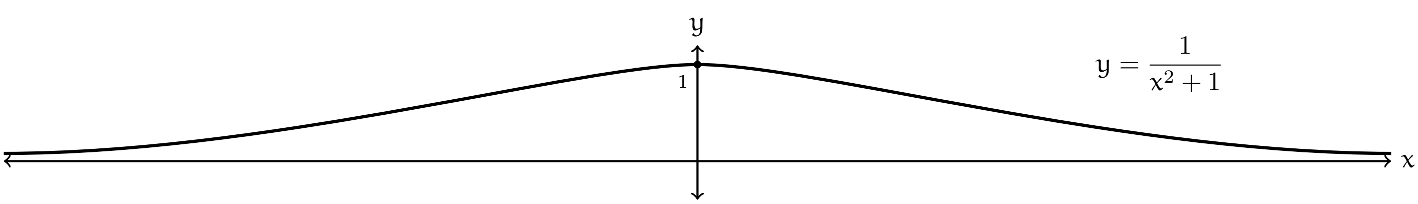
- $\lim_{x\to\infty} f(x) = L$ means that x approaching ∞ causes $f(x)$ to approach the number L.

  In such a case the line $y = L$ is a horizontal asymptote.

- $\lim_{x\to-\infty} f(x) = M$ means that x approaching $-\infty$ causes $f(x)$ to approach the number M.

  In such a case the line $y = M$ is a horizontal asymptote.

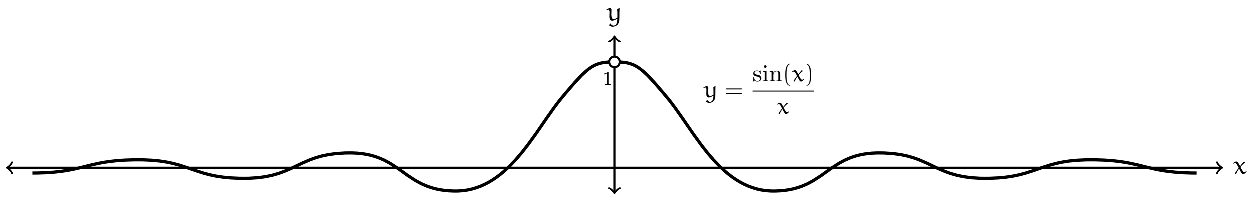
For a concrete example, consider the function $f(x) = \dfrac{1}{x^2+1}$. As x approaches ∞, the denominator also approaches ∞, but the numerator remains 1. Thus the quotient becomes smaller and smaller, approaching zero. Therefore $\lim_{x\to\infty} f(x) = 0$, so the line $y = 0$ (which is the x-axis) is a horizontal asymptote. Similarly, as x approaches $-\infty$, the denominator once again approaches ∞, while the numerator is 1, and again the quotient approaches 0. Therefore $\lim_{x\to-\infty} f(x) = 0$, and we get the same horizontal asymptote $y = 0$. This function is graphed below. It has only one horizontal asymptote.



For many functions, such as $\sin(x)$ and x^2, the limits at infinity do not exist, so they have no horizontal asymptotes at all. For example, consider $\lim_{x\to\infty} \sin(x)$. As x approaches ∞, the function $\sin(x)$ just oscillates between 1 and -1, and does not approach any one number. Therefore $\lim_{x\to\infty} \sin(x)$ does not exist, and there is no horizontal asymptote (as the graph of $\sin(x)$ confirms). For the function x^2, we could say $\lim_{x\to\infty} x^2 = \infty$, but it does not make sense for the horizontal asymptote to be the "line" $y = \infty$ because there is no such line on the Cartesian plane. The function x^2 has no horizontal asymptote.

As the above examples suggest, a given function may have one horizontal asymptote, two horizontal asymptotes, or none at all. But no function can have more than two horizontal asymptotes, because there are only two directions in which x can approach infinity – either in the positive or negative directions.

It is possible for a graph to cross its horizontal asymptotes. Consider the function $f(x) = \frac{\sin(x)}{x}$, graphed below. As the numerator $\sin(x)$ is between -1 and 1 for any x, the quotient $\frac{\sin(x)}{x}$ is very small for large values of x. Then $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0 = \lim_{x\to-\infty} \frac{\sin(x)}{x}$, so the x-axis $y = 0$ is a horizontal asymptote. The graph of $y = f(x)$ oscillates above and below this line, squeezing in on it as x gets big. It crosses the asymptote at each point $x = k\pi$, at which $\sin(x) = 0$.



A function's limits at infinity and horizontal asymptotes are significant because they give potentially useful information about how the function behaves for very large values of x. Get lots of practice evaluating limits of form $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and finding horizontal asymptotes.

How to Evaluate Limits at Infinity

There is no single method for doing this, and it often requires some artistry. Here are some common tricks. (They work for $x \to -\infty$ as well as $x \to \infty$.)

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is such that $g(x)$ is approaching ∞ and $f(x)$ is bounded, then $\lim_{x\to\infty} \frac{g(x)}{h(x)} = 0$.
- In $\lim_{x\to\infty} \frac{f(x)}{g(x)}$, identify (if possible) the highest power x^n that appears in $\frac{f(x)}{g(x)}$, and multiply by 1 in the form $\frac{1/x^n}{1/x^n}$. Distribute into the fraction and exploit the (easy) limit $\lim_{x\to\infty} \frac{1}{x^k} = 0$.
- If $\frac{f(x)}{g(x)}$ is a rational function (i.e., a polynomial divided by a polynomial) then:
  - If the degree of $f(x)$ is greater than the degree of $g(x)$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$. (sign determined by inspecting $g(x)$ and $h(x)$).
  - If the degree of $f(x)$ is less than the degree of $g(x)$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
  - If $f(x)$ and $g(x)$ have the same degree, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$, where a is the coefficient of the highest power of x in $f(x)$ and b is the coefficient of the highest power of x in $g(x)$.

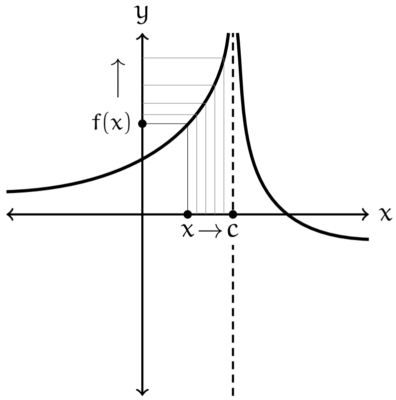
If $\lim_{x\to\infty} f(x) = L$, then it is very often true that also $\lim_{x\to-\infty} f(x) = L$. This is always the case for rational functions. But occasionally (especially when radicals are involved) it may happen that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$ with $L \neq M$.

How to find the horizontal asymptotes (if any) of f(x)

1. Compute $\lim_{x\to\infty} f(x)$. If you get a finite number L then the line $y = L$ is a horizontal asymptote.
2. Compute $\lim_{x\to-\infty} f(x)$. If you get a finite number M then the line $y = M$ is a horizontal asymptote.

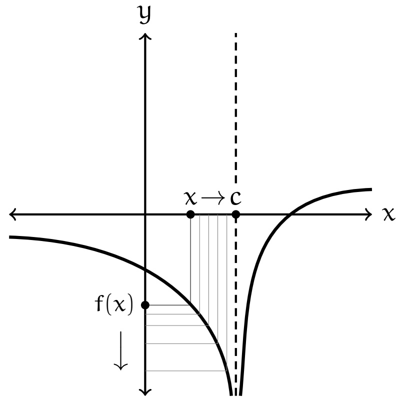
Infinite Limits and Vertical Asymptotes

Sometimes a limit $\lim_{x\to c} f(x)$ does not exist, but it does not exist for a very particular reason: as x approaches c, the corresponding value $f(x)$ does not approach any number, but just gets larger and larger, without bound. This is illustrated in the left-most graph below. Roughly speaking, the function $f(x)$ "blows up" at $x = c$. We express this symbolically as $\lim_{x\to c} f(x) = \infty$, which is called an **infinite limit**. It means that as x approaches c, the value $f(x)$ becomes larger and larger (in the positive direction). In such a case the graph of $y = f(x)$ becomes closer and closer to the vertical line $x = c$ passing through the point c on the x-axis. We say that the line $x = c$ is a **vertical asymptote** of $f(x)$.



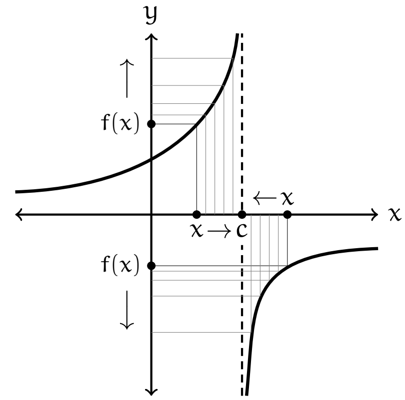
$\lim_{x\to c} f(x) = \infty$

Line $x = c$ is a V.A.



$\lim_{x\to c} f(x) = -\infty$

Line $x = c$ is a V.A.



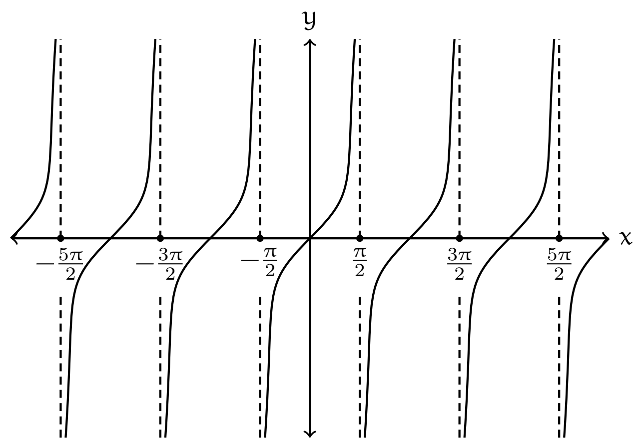
$\lim_{x\to c} f(x)$ DNE

$\lim_{x\to c^-} f(x) = \infty$

$\lim_{x\to c^+} f(x) = -\infty$

In the middle graph above, as x approaches c the value $f(x)$ becomes bigger and bigger in the negative direction. We express this as $\lim_{x\to c} f(x) = -\infty$, and again the line $x = c$ is a vertical asymptote. Sometimes right- and left-hand limits are needed to express the behavior of $f(x)$ near $x = c$. For the function graphed above on the right we have $\lim_{x\to c^+} f(x) = -\infty$ and $\lim_{x\to c^-} f(x) = \infty$.

Although a function never has more than two *horizontal* asymptotes, it can have any number of *vertical* asymptotes (or none at all, depending on the function). The function $f(x) = \tan(x)$, graphed to the right, has infinitely many vertical asymptotes. For each odd integer k, the line $y = \frac{k\pi}{2}$ is a vertical asymptote.



However, a great many functions (such as polynomials and the trigonometric functions $\sin(x)$ an $\cos(x)$) have no vertical asymptotes at all. But when they are present, vertical asymptotes help us understand certain properties of a function. Knowing that the line $x = c$ is a vertical asymptote tells us that even though $f(c)$ may not be defined at $x = c$, the function $f(x)$ "blows up" near c.

Typically a function $f(x)$ will have an infinite limit (and hence also a vertical asymptote) at $x = c$ when it has the form $f(x) = \frac{g(x)}{h(x)}$ and $h(c) = 0$. This is the case with $\tan(x) = \frac{\sin(x)}{\cos(x)}$, graphed above; the vertical asymptotes $x = \frac{k\pi}{2}$ happen at precisely the values of x that make the denominator $\cos(x)$ equal to zero.

In working out a limit $\lim_{x\to c} \dfrac{g(x)}{h(x)}$, we usually don't know ahead of time that it is going to be infinite. We use our usual procedure for evaluating such a limit, outlined on the first page of this sheet. If we find that $\lim_{x\to c} h(x) = 0$, but we cannot cancel a factor that is causing $h(x)$ to approach zero, then we may begin to suspect that the limit is going to be infinite. To find out for sure, look at $\lim_{x\to c} g(x)$. If this is not zero then $\lim_{x\to c} \dfrac{g(x)}{h(x)}$ involves a numerator that approaches a non-zero number and a denominator that is approaching zero. Therefore the limit will be ∞ or $-\infty$. The sign can determined by analyzing the signs of $g(x)$ and $h(x)$ for x values near c.

How to find the vertical asymptotes (if any) of f(x)

1. Identify the values $x = c$ that make the denominator of $f(x)$ equal to zero or that make $f(x)$ undefined.
   These are the **candidates** for the locations of the vertical asymptotes.
2. For each c obtained in the previous step, evaluate $\lim_{x\to c^+} f(x)$ or $\lim_{x\to c^-} f(x)$.
   If you get $\pm\infty$, then the line $x = c$ is a vertical asymptote.