# Probabilistic proofs of hook length formulas involving trees 

by

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Let $T$ be a rooted tree with $n$ distinguishable vertices. We use $T$ to stand for the vertex set of $T$. An increasing labeling of $T$ is a bijection $\ell: T \rightarrow\{1,2, \ldots, n\}$ such that $\ell(v) \leq \ell(w)$ for all descendents $w$ of $v$. Let $f^{T}$ be the number of increasing labelings. The hooklength, $h_{v}$, of a vertex $v$ is the number of descendents of $v$ (including $v$ itself). The hook length formula for trees states that

$$
f^{T}=\frac{n!}{\prod_{v \in T} h_{v}} .
$$

There is a similar formula for the number of standard Young tableaux of given shape where a hooklength is the cardinality of a set which resembles a physical hook. Greene, Nijenhuis, and Wilf gave a beautiful probabilistic proof of the tableau formula where the hooklenths enter in a very natural way.

Recently, Han discovered a formula which has the interesting property that hooklengths appear as exponents. Specifically, let $\mathcal{B}(n)$ be the set of all $n$-vertex binary trees (each vertex has no children, a left child, a right child, or both children). Han proved that

$$
\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{1}{h_{v} 2^{h_{v}-1}}=\frac{1}{n!}
$$

using algebraic manipulations. We will show how to give a simple probabilistic proof of this equation as well as various generalizations. We will also pose some open questions raised by this work.

