Vizing's Conjecture: 4 decades later

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A parameter σ is called *submulitiplicative* on a graph product $\otimes$ if $\sigma(G \otimes H) \leq \sigma(G)\sigma(H)$ for all graphs G, H. A parameter σ is called *supermulitiplicative* on a graph product $\otimes$ if $\sigma(G \otimes H) \geq \sigma(G)\sigma(H)$ for all graphs G, H. A subset of vertices $S \subseteq V(G)$ is *dominating* if every vertex in $V(G)$ is either in S or adjacent to a vertex in S. The cardinality of the smallest dominating set is denoted by $\gamma(G)$, the domination number. The *Cartesian product* of G and H is denoted by $G \Box H$; the vertices are the ordered pairs $\{(x, y) | x \in V(G), y \in V(H)\}$. Two vertices (u, v) and (x, y) are adjacent in the Cartesian product if and only if one of the following is true: $u = x$ and v is adjacent to y in H; or $v = y$ and u is adjacent to x in G.

V.G. Vizing conjectured that for all graphs G and H,

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H).$$

That is, Vizing conjectured that γ was supermultiplicative on the Cartesian product. This conjecture has remained open since 1968. In contrast, it is not too hard to see why a variant of the domination number, the total domination number, is submultiplicative on the direct product.

A subset $S \subseteq V(G)$ is a 2-*packing* if the minimum distance between any distinct vertices u, v of S is at least three. The packing number $\rho(G)$ is the size of a largest 2-packing. It is known that Vizing's conjecture will hold for every graph H, if G satisfies at least one of the following: $\gamma(G) \leq 3$; $|\gamma(G) - \rho(G)| \leq 1$; G is chordal; G is the a spanning subgraph of K with $\gamma(G) = \gamma(K) = \chi(\overline{K})$. In this talk, we outline failed attempts to construct a counter-example to Vizing's conjecture, and in the process establish a lower bound on the number of vertices of a counterexample (should one exist).

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