ELSEVIER

Economics Letters 50 (1996) 367–371

economics letters

A variant of the Slutsky equation in a dynamical account based model

Hassan Sedaghat

Department of Mathematical Sciences, Virginia Commonwealth University, Richmond, VA 23284-2014, USA

Received 12 May 1995; accepted 23 August 1995

Abstract

Using the movements caused by the variations of income or prices of a dynamically stable account fixed point, it is possible to define an observable demand function that satisfies a variant of Slutsky's equation.

Keywords: Consumer account; Stable fixed point; Demand function

JEL classification: D11; D91

1. Introduction

Ever since its appearance in the work of Allen and Hicks (1934), the familiar equation that dates back to Slutsky (1915) has served as a fundamental relationship in consumer theory. However, this equation is derived in the context of the static preference maximization model and is not particularly amenable to dynamical considerations (for instance, see Chaudhuri, 1995, who also discusses some of the alternatives to the Slutsky equation). Furthermore, current intertemporal models typically use the device of discounted utility – a concept that is economically difficult to work with.

In this paper we derive a variant of the Slutsky equation in a dynamical model that is based on the way the consumer's behavior is reflected in a consumer account. The concept of utility is not required in any form, so we avoid some of the complications that arise from the use of that notion. Since this brief paper is primarily intended to show the feasibility of a dynamical non-utility approach, we discuss a restricted model in which only income (mainly wages) and general commodity consumption is considered.

2. Consumer accounts and stability

Instead of a preference relation on the consumption bundles, we make use of an accounting identity and its index functions. These concepts pertain to a *consumer account* that is not

SSDI 0165-1765(95)00740-7

unlike the Keynesian national income accounts with the *account balance* in place of the 'output'. Of course, consumer accounts are economically significant only in bulk, and thus only their *stable qualitative behavior* is of interest.

Let $\mu \geq 0$ denote the disposable income, $q^t = (q_1^t, \ldots, q_m^t)$ the vector of quantities consumed in period t, and $p = (p_1, \ldots, p_m)$ the corresponding vector of unit prices of the m goods involved. Then

$$x_t - x_{t-1} = \mu - p \cdot q^t\,, \quad t = 1, 2, 3, \ldots, \tag{1}$$

where x_t denotes the amount of money remaining in the account at the *end* of period t (i.e. the account balance) and $p \cdot q^t = \Sigma_{j=1}^m p_j q_j^t$ represents the total expenditures. The constancy of income and prices in Eq. (1) reflects their parameter-like, exogenous nature and our intent to vary them in a comparative static sense. With regard to q^t, we adopt the following simplifying hypothesis:

(H1) $q^t = q(x_{t-1})$ depends only on the balance of money at the beginning of period t.

The use of this hypothesis restricts the range of goods to those that are consumed within one period (recorded consumption – e.g. the sum of real money spent on food in one period; say, a month – may differ from physical consumption, which may occur several times within a single period). Since prices and income do vary, periods cannot be of arbitrarily long duration, so our focus here is mainly on non-durable or flow goods. Though not discussed here, a weaker form of (H1), in which loan payments and explicit time dependence in the functional form q are allowed, not only permits the consideration of durable goods but also allows needs or tastes to change in time (e.g. seasonal variations in the consumption of some goods, such as heating gas).

We call the function $q(x)$ the *quantity index* of the account and define $E_p(x) = p \cdot q(x)$ as the *expenditure index* of the account. We assume in this paper that q is continuously differentiable. An *account fixed point* is a solution x^* of the equation $\mu = E_p(x)$. Such a solution clearly exists if $E_p^- < \mu < E_p^+$ where

$$E_p^- = \inf_{-\infty<x<\infty} E_p(x) \geq 0\,, \qquad E_p^+ = \sup_{-\infty<x<\infty} E_p(x) \leq \infty\,,$$

and x^* is also unique if $E_p(x)$ is an increasing function in x. Conditions for the global asymptotic stability of x^* are stated in Sedaghat (1995), while local asymptotic stability conditions may be derived from the standard linear stability results (see, for example, Day, 1994).

3. Demand functions

Call a price vector p in the positive orthant $(0, \infty)^m$ *regular* if E_p is increasing everywhere. We assume that

(H2) There are regular price vectors.

The implications of (H2) are discussed in detail in Sedaghat (1995), where it is argued that under reasonable circumstances the set Ψ of all regular price vectors may be taken for all practical purposes to be an open m-rectangle $\Pi_{i=1}^{m}(a_i, b_i)$, where $0<a_i<b_i<\infty$, $i=1, 2, \ldots, m$. If such a Ψ is sufficiently small in volume and does not contain the origin of its boundary, then the set

$$\Omega = \bigcap_{p\in\Psi} (E_p^-, E_p^+) \subset (0,\infty)$$

has a non-empty interior. The function $D:\Omega\times\Psi\to[0,\infty)^m$, defined by

$$D(\mu, p) = q(x^*) = q(E_p^{-1}(\mu)) ,$$

is a demand function of income and prices the jth coordinate D_j of which gives the consumption of the jth good at the equilibrium account level $x^* = x^*(\mu, p)$. Since the budget identity $\mu = p\cdot q(x^*)$ does not change by the multiplication of a constant on both sides, it can be easily shown that D is homogeneous of degree zero over its domain.

Since E_p is increasing for every regular p, a unique fixed point always obtains and is given by the expression $E_p^{-1}(\mu)$. We further assume (by taking subsets if necessary) that x^* is at least locally asymptotically stable for all pairs in $\Omega\times\Psi$. If either the income or the prices change within the domain of D, then so does $x^* = x^*(\mu, p)$. If x_1^* is the old fixed point and x_2^* is the fixed point after (μ, p) changes, then in the globally stable case the account trajectory $\{x_t\}$ moves from x_1^* towards x_2^*, and after a transitional period, T_{12}, the trajectory is always close enough to x_2^* to be virtually indistinguishable from x_2^* (in the locally stable case, this happens if x_2^* and x_1^* are sufficiently close to one another). T_{12} may be defined relative to a fixed tolerance ε as

$$T_{12} = \min\{t\geq 0: |x_2^* - x_t - \mu_2 + E_{p_2}(x_t)| < \varepsilon\} , \quad x_0 = x_1^* ,$$

where μ_2 and p_2 denote the new values of income and prices, respectively. Thus, stability conditions result in dynamically observable values of D within the tolerance ε.

Theorem. *Assume that $\Omega\times\Psi$ has a non-empty interior and that for all (μ, p) in the interior of $\Omega\times\Psi$, the unique fixed point is asymptotically stable. Then D is well defined with observable values on the interior of $\Omega\times\Psi$, and it satisfies the matrix equation:*

$$\frac{\partial D_j}{\partial p_i} = -D_i \frac{\partial D_j}{\partial \mu} , \quad i, j = 1, 2, \ldots, m . \tag{2}$$

Furthermore, the fixed point x^* satisfies the numerical differential identity:

$$E_p'(x^*(\mu, p))\,\mathrm{d}x^* = \mathrm{d}\mu - D(\mu, p)\cdot \mathrm{d}p , \quad (\mu, p)\in\Omega\times\Psi . \tag{3}$$

Proof. By the definitions of Ω and Ψ, and the hypotheses of the theorem, $E_p'(x) > 0$ for all x, p and a unique, stable fixed point $x^*(\mu, p)$ always exists for all (μ, p) in the interior of $\Omega \times \Psi$. Upon differentiating the budget identity

$$\sum_{k=1}^{m} p_k q_k(x^*(\mu, p)) = E_p(x^*(\mu, p)) = \mu$$

with respect to p_i, we obtain

$$q_i(x^*) + \frac{\partial x^*}{\partial p_i} \sum_{k=1}^{m} p_k \frac{\mathrm{d}q_k}{\mathrm{d}x}(x^*) = 0 .$$

Solving the last equality for $\partial x^*/\partial p_i$ we find

$$\frac{\partial x^*}{\partial p_i} = \frac{-q_i(x^*)}{\sum_{k=1}^{m} p_k \, \mathrm{d}q_k/\mathrm{d}x(x^*)} = \frac{-D_i}{E_p'(x^*)} . \tag{4}$$

Now, using (4) and the chain rule gives

$$\frac{\partial D_j}{\partial p_i} = \frac{\mathrm{d}q_j}{\mathrm{d}x}(x^*) \frac{\partial x^*}{\partial p_i} = \frac{-D_i \, \mathrm{d}q_j/\mathrm{d}x(x^*)}{E_p'(x^*)} . \tag{5}$$

Since $x^* = E_p^{-1}(\mu)$ and $D_j = q_j \circ x^*$, we find

$$\frac{\partial D_j}{\partial \mu} = \frac{\mathrm{d}q_j}{\mathrm{d}x}(x^*) \frac{\partial E_p^{-1}}{\partial \mu} = \frac{\mathrm{d}q_j}{\mathrm{d}x}(x^*) \frac{1}{E_p'(x^*)} ,$$

which together with (5) yields the matrix equation (2). To prove (3), note that using Eq. (4):

$$\mathrm{d}x^* = \nabla x^*(\mu, p) \cdot (\mathrm{d}\mu, \mathrm{d}p) = \frac{\partial x^*}{\partial \mu} \mathrm{d}\mu - \frac{D(\mu, p)}{E_p'(x^*(\mu, p))} \cdot \mathrm{d}p ,$$

where $(\mathrm{d}\mu, \mathrm{d}p)$ represents the vector of increments in prices and income. Using the fact that $\partial x^*/\partial \mu = 1/E_p'(x^*)$, the equation for $\mathrm{d}x^*$ may be written in the more succinct form (3). □

The matrix equation (2) looks like the Slutsky equation without the compensated demand term; this is natural since we do not use utility functions. This similarity is remarkable, considering the completely different conceptual and mathematical settings in which each equation is derived. However, (2), which is equivalent to (5), is *not* a special case of the Slutsky equation, since it involves the quantity and expenditure index functions that do not appear in the Slutsky equation. These index functions make up for the missing compensated demand function after we translate the familiar classification of goods (e.g. normal, inferior) into properties for the index functions q_j; the rather extensive details appear in Sedaghat (1995).

As an example, if the first good is normal, then its quantity index curve q_1 can be shown to be increasing as a function of x. Hence $\mathrm{d}q_1/\mathrm{d}x > 0$, which results in $\partial D_1/\partial p_i$ being negative in this case for all i. Using reasoning of this type, we may seek conditions that imply negative semidefiniteness for the matrix on the right side of (5); however, in many applications it may be easier (and preferable from the dynamical viewpoint) to use the *numerical* equation (3) instead; for instance, to prove a generalization of the classical demand law, note that if $d\mu = 0$ and $dp_i \geqslant 0$ for all i with strict inequality holding for at least one i, then $dx^* < 0$. With the first good normal, q_1 is increasing, so a reduction in x^* results in a reduction in consumption $q_1(x^*)$.

Acknowledgements

I wish to thank Professor William Baumol for his valuable and encouraging remarks on the approach of this paper.

References

Allen, R.G.D. and J.R. Hicks, 1934, A reconsideration of the theory of value, Economica 14, I: 2–76, II: 196–219.
Chaudhuri, A., 1995, On the relationship between the Frisch and Slutsky decompositions, Economics Letters 47, 283–290.
Day, R.H., 1994, Complex economic dynamics, vol. 1 (MIT Press, Cambridge, MA).
Sedaghat, H., 1995, A new approach to consumer demand (preprint).
Slutsky, E., 1915, Sulla teoria del bilancio del consomatore, Giornale degli Economisti 51, 1–26.