Thresholds, Mode Switching and Complex Dynamics

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Abstract In this note, mode switching in threshold systems is considered in the special case of ejector cycles. The existence and the global properties of such a cycle (from periodic behavior to chaos) are studied in a model from combat theory.

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1 Introduction

Systems of difference equations that possess thresholds and discontinuities are quite commonly encountered in scientific models in general, and models in the social sciences in particular; see, e.g., [1-4] and the references therein. A few specific examples appear below. In spite of this, there are few mathematical tools that are broadly applicable to these systems. In this note, we consider briefly first a general classification of threshold models as polymodal systems, and then consider the concept of *ejector cycles* that pertains to such systems. These cycles contain global information about the system as each goes through its mode switching sequence. We illustrate this behavior by analyzing equations from a model of ground combat.

2 Polymodal systems

Defintion 1 *Let $D \subset \mathbb{R}^m$ be a nonempty set, and let $F \in C(D, \mathbb{R}^m)$. A point $\widehat{x} \in D$ is an* ejection point *of F if $F(\widehat{x}) \notin \overline{D}$. The set E of all ejection points of F in D is the* ejector of D relative to F*; i.e.,*

$$E \doteq \left\{x \in D : F(x) \notin \overline{D}\right\}.$$

Defintion 2 *A* polymodal system in $\mathbb{R}^m$ with k modes *is a function-set collection*

$$\{(F_i, D_i) : i = 1, \ldots, k\}, \; k \geq 2$$

where for each i all of the following are true:

(a) $D_i \subset \mathbb{R}^m$ is nonempty and disjoint from D_j for $j \neq i$;
(b) $F_i \in C(D_i, \overline{D})$ where $D \doteq \bigcup_{j=1}^{k} D_j$;
(c) D_i contains a nonempty ejector E_i relative to F_i.

Each pair (F_i, D_i) is a *mode* of the system. We also define the usually, though not necessarily, discontinuous *join* of the maps F_i as

$$F \doteq \sum_{i=1}^{k} \chi_{Di} F_i : D \rightarrow \overline{D},$$

where χ_S is the characteristic function of the set S, i.e., $\chi_S(x) = 1$ if $x \in S$ and $\chi(x) = 0$ if $x \notin S$.

Next, we give some examples of polymodal systems from the social sciences. For additional examples, and a more detailed discussion of polymodal systems, see [6].

Examples 1 (Addiction, duopoly, arms race) G. Feichtinger proposes two sets of equations in [4] (and the references given therein) that involve thresholds (and are thus polymodal). The first set:

$$\begin{aligned} x_{n+1} &= a x_n + b \chi_{\{x_n > y_n\}} \\ y_{n+1} &= y_n + c(x_n - y_n) \end{aligned}$$

models "habit formation" in use of addictive substances, (e.g., tobacco, alcohol, drugs) where x_n is the habit's (e.g., smoking) consumption capital in period n and y_n is the "threshold in the habit stock" so that consumption takes place only if x_n exceeds y_n. Also, $0 < a < 1$, $b, c > 0$. It is easy to see that this is a bimodal system with D_1 and D_2 the opposite sides of the diagonal $y = x$.

The second set of equations model dynamic interaction in a simple duopoly with "asymmetries." If x_n and y_n denote the sizes (as measured by sales or market shares) of the two firms in the duopoly in period n, then

$$\begin{aligned} x_{n+1} &= (1-\alpha) x_n + a \chi_{\{x_n > y_n\}} \\ y_{n+1} &= (1-\beta) x_n + b \chi_{\{x_n > y_n\}} \end{aligned}$$

where $\alpha, \beta \in (0, 1)$ and $a, b > 0$. These equations may also be used as a nonlinear extension of Richardson-type model of arms race.

Example 2 (A model of ground combat) J. Epstein [3] proposes a simple deterministic model in order to illustrate the role of a defender's withdrawal as a feedback mechanism that can substantially affect the outcome of combat. A special case of this model involves the following equations:

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{a}(a - x_n)\left[a - x_n(1 - y_n)\right] \\ y_{n+1} &= \left\{ y_n + \frac{1 - y_n}{1 - d}[x_n(1 - y_n) - d] \right\} \chi_{\{x_n(1-y_n) \geq d\}} \end{aligned} \tag{1}$$

These equations are taken from [5] where the ground-combat version of Epstein's model is treated in a rigorous way. There are two combatants, an "attacker" and a "defender". The latter will withdraw if its attrition level exceeds a prescribed level (losses measured in terms of standard military "scores"). The variables and constants have the following meanings:

a: attacker's attrition rate threshold, $a \in (0, 1)$
d: defender's attrition rate threshold, $d \in (0, 1)$
y_n: defender's withdrawal rate in period (e.g., day) n
x_n: attacker's prosecution rate of combat in period n
$x_n(1 - y_n)$: defender's attrition rate

Also, assumed are the initial value restrictions: $x_0 > 0$, $y_0 = 0$. Equations (1) describe a threshold model, which can be expressed as a bimodal system:

$$\begin{aligned} D_1 &= \{(x, y) \in [0, \infty)^2 : x(1 - y) \geq d\} \\ D_2 &= \{(x, y) \in [0, \infty)^2 : x(1 - y) < d\} \\ F_1(x, y) &= [f(x, y), g(x, y)], \ (x, y) \in D_1 \\ F_2(x, y) &= [f(x, y), 0], \ (x, y) \in D_2 \end{aligned}$$

where

$$\begin{aligned} f(x, y) &= x + (a - x)[a - x(1 - y)]/a \\ g(x, y) &= y + (1 - y)[x(1 - y) - d]/(1 - d). \end{aligned}$$

Sets D_1 and D_2 represent the regions in $[0, \infty)^2$ that lie, respectively, below and above the curve $y = 1 - d/x$.

Examples 3 (One dimensional, multi-regime economic systems) Economic models often involve thresholds that range from physical constraints (since negative values are not permissible either for functions or for variables) to different "regimes" that a system can exists in. Piecewise linear equations provide the simplest examples of these types of systems. Several polymodal models are discussed in [2] together with a discussion of these models as "multiple phase systems." In all the polymodal systems in [2] the join map $\sum_{i=1}^{k} \chi_{D_i} F_i$ is continuous (though not necessarily smooth). More economic examples (too many to list specifically) appear in other sources, e.g., [1].

3 Ejector cycles and an application

Each ejector E_i by itself is anti-invariant, where points of D_i move out by the action of F. In some cases, a collection of two or more ejectors can form an *ejector chain* that leads trajectories through the sets D_i. When such a chain is closed, trajectories return to the ejector from which they started, and the initial ejector will contain an invariant subset under the action of several of the F_i composed with each other.

Defintion 3 *Let (F_i, D_i), $i = 1, \ldots, k$, $k \geq 2$ be a polymodal system with k components. Let $2 \leq l \leq k$ and assume that $E_{i_1}, \ldots, E_{i_l}$ are ejectors in $D_{i_1}, \ldots, D_{i_l}$ respectively. Then the collection $\mathcal{E} = \{E_{i_1}, \ldots, E_{i_l}\}$ is an* ejector cycle *of length l if for each $j = 1, \ldots, l$ there is a nonempty subset $E'_{i_j} \subset E_{i_j}$ such that*

$$F_{i_j}(E'_{i_j}) \subset E'_{i_{j+1}}, \ 1 \leq j \leq l-1, \ F_{i_l}(E'_{i_l}) \subset E'_{i_1}.$$

We call the continuous mapping

$$\psi \doteq F_{i_l} \circ \cdots \circ F_{i_1} : E'_{i_1} \to E'_{i_1}$$

a cycle map *of the polymodal system corresponding to the ejector cycle $\mathcal{E}$.*

Note that a cycle map is a standard continuous mapping on an invariant region, namely E'_{i_1}. As such, the standard theory of continuous maps applies to ψ and the various properties of ψ provide information on the behavior of the polymodal system. For example, if ψ has a cycle of length l, then there is a cycle of length kl in the polymodal system. Similarly, an aperiodic trajectory of ψ gives rise to an aperiodic trajectory in the system. It is therefore of great interest to identify ejector cycles, whenever they exist.

It is not difficult to give specific examples of polymodal systems in dimensions 1 and 2 that possess ejector cycles and exhibit complex behavior. Here, however, we study the system described by equations (1) above. Our aim is to show that for various ranges of parameter values, *there are ejector cycles whose cycle maps are topologically conjugate to maps of the interval.* This fact is then used to draw conclusions about the behavior of equations (1). We begin with two lemmas whose proofs are given in [5].

Lemma 1 *Let ξ be the largest real root of the cubic polynomial*

$$C(t) = -(1-t)(t^2 - at + a^2) + ad(1-d).$$

Then $\xi < 1$. If $a > 1/2$, then C is strictly increasing and ξ is its only real root. Further, $\xi \in (d, a)$ if $a > d \geq 1/2$ and $\xi \in (1-d, a)$ if $1-a < d < 1/2$.

Lemma 2 *Let $d \leq a$ and consider the quintic polynomial*

$$Q(t) = a + t(1-t)(t-a)\frac{t^2 - at + a^2}{a^3(1-d)}, \quad t \geq 0.$$

(a) *Q has a unique real root ζ, and $\zeta \in (1, 1+a)$; in fact, there is $\varepsilon > 0$ such that Q is strictly decreasing on the interval $(1-\varepsilon, \infty)$, and Q maps the interval $[1, \zeta]$ homeomorphically onto $[0, a]$;*

(b) *Assume that $d < a$. Then all fixed points of Q that exceed d are in the interval $[a, 1)$. If $a \geq 1/2$, then a is the only fixed point of Q that is larger than d. On the other hand, if $a < 1/2$ and*

$$d \geq 1 - \frac{1}{4a} \tag{2}$$

then Q has a fixed point in (a, β^-) and another in $(\beta^+, 1)$, where

$$\beta^{\pm} = \frac{1 \pm \sqrt{1 - 4a(1-d)}}{2}.$$

Theorem 1 (a) *Let $1 - a < d < 1/2$ and let $\xi \in (1-d, a)$ be the unique zero of $C(t)$ in Lemma 1. If $\{(x_n, y_n)\}$ is a solution of (1) with $y_0 = 0$ and $x_0 \in (\xi, a]$ then $\{x_n\}$ increases monotonically to a, but for all n,*

$$y_{2n} = 0, \;\; y_{2n+1} = \frac{x_{2n} - d}{1 - d}$$

In particular, the trajectory $\{(x_n, y_n)\}$ converges to the 2-cycle

$$\Gamma = \{(a, 0), (a, y_\infty)\}$$

where $y_\infty = (a - d)/(1 - d)$.

(b) *Let $a > d \geq 1/2$, and let $\xi \in (d, a)$ be the unique zero of $C(t)$ in Lemma 1. Then the same behavior as in Part (a) is obtained.*

Proof. (a) We first show that system (1) has an ejector cycle. For $x > \xi$,

$$\begin{aligned}
\psi(x, 0) &= F_2 \circ F_1(x, 0) \\
&= F_2(x + (a-x)^2/a, (x-d)/(1-d)) \\
&= (Q(x), 0)
\end{aligned}$$

where Q is defined in Lemma 2. The action of F_2 is well defined because if $\tilde{x}_1$ and $\tilde{y}_1$ denote the two coordinates of $F_1(x, 0)$ above, then $\tilde{x}_1(1 - \tilde{y}_1) < d$ if and only if $C(x) > 0$. This last inequality is true by Lemma 1 if $x > \xi$. Further, if $x \in (\xi, a]$, then

$$a - Q(x) = \frac{\tilde{x}_1(1 - \tilde{y}_1)}{a}(a - \tilde{x}_1) < a - \tilde{x}_1$$

so that

$$Q(x) > \tilde{x}_1 > x. \tag{3}$$

Since $Q(a) = a$, it follows that $Q(I) \subset I$ where $I = (\xi, a]$. Hence, ψ is the cycle map of an ejector cycle with domain $E_1' = I \times \{0\}$, and ψ is topologically isomorphic to Q on I. In addition, the inequalities (3) imply that $\{x_n\}$ increases to a if $x_0 \in (\xi, a]$ (the even terms are $x_{2n} = Q^n(x_0)$ and the odd terms are given by $x_{2n+1} = \tilde{x}_1(x_{2n})$). Also, $\{y_n\}$ behaves as claimed, since $y_{2n} = 0$ (being the second coordinate of $F_2 \circ F_1(x_{2n-2}, 0)$) and $y_{2n+1} = (x_{2n} - d)/(1-d)$, which converges to y_∞.

(b) This is done in essentially the same way as (a). ■

Remark The image $F_1(I, 0)$ of I is the locus of all odd terms (x_{2n+1}, y_{2n+1}). These are the coordinates of $F_1(x_{2n}, 0)$, i.e.,

$$x_{2n+1} = x_{2n} + (a - x_{2n})^2/a, \quad y_{2n+1} = (x_{2n} - d)/(1-d).$$

Eliminating x_{2n} from these equations shows that $F_1(I, 0)$ is a connected segment of the parabola

$$x = l(y) = d + (1-d)y + \frac{1}{a}[a - d - (1-d)y]^2. \tag{4}$$

Theorem 2 below looks at the situation where $x_0 > a$. Again, we find an ejector cycle, but the dynamics are not as simple as in Theorem 1.

Theorem 2 *Assume that one of the following conditions hold:*

(i) $a > d \geq 1/2$;

(ii) $1 - a < d < 1/2$.

Then every trajectory with $x_0 \in (a, 1)$ and $y_0 = 0$ converges to the cycle Γ of Theorem 1 from the right, with $\{y_n\}$ having the same behavior as in Theorem 1(a) but now $\{x_n\}$ converges non-monotonically to a from the right in the manner $x_{2n+2} < x_{2n} < x_{2n+1}$ for every n.

Proof. As in the proof of Theorem 1, we have

$$\psi(x, 0) = F_2(\tilde{x}_1, \tilde{y}_1) = (Q(x), 0). \tag{5}$$

If (i) or (ii) hold, and ξ is as defined in Lemma 1, then $\xi < a$ so $\tilde{x}_1(1 - \tilde{y}_1) > d$, and the action of F_2 is well defined for $x > a$. Also by Lemma 2, Q has no fixed points (except a). It follows that if $x \in (a, 1)$ and $y = 0$, then $\tilde{x}_1 > x$. Further,

$$0 < Q'(a) = \frac{1-a}{1-d} < 1$$

so that $a < Q(x) < x < \tilde{x}_1$ for $x \in (a, 1)$. These inequalities and (5) establish the pattern described in the statement of the theorem. ■

As in the previous case, it is evident that the odd terms (x_{2n+1}, y_{2n+1}) fall on a connected segment of the parabola (4) and the even terms (x_{2n}, y_{2n})

fall in the interval $I = (a, 1)$ on the x-axis. We quote one more result that gives sufficient conditions for the occurrence of chaotic behavior. The proof is self-evident, since ψ is topologically conjugate to Q on a suitable interval I.

Theorem 3 *Assume that the polynomial Q has a fixed point in the interval $(a, 1)$, and that p is the larger fixed point with $p > \xi$. If the interval $(\xi, 1)$ contains a subinterval I with $p \in Q(I) \subset I$, and if $x_0 \in I$, $y_0 = 0$, then for $n \geq 1$, the following are true:*

(a) *If p is attracting (e.g., $|Q'(p)| < 1$), then $\{(x_n, y_n)\}$ converges to the 2-cycle*

$$\Psi = \left\{(p, 0), \left(p + \frac{(p-a)^2}{a}, \frac{p-d}{1-d}\right)\right\}.$$

(b) *If p is unstable (e.g., $Q'(p) < -1$), and Q has a limit cycle $\{c_1, \ldots, c_k\}$ in I, then $\{(x_n, y_n)\}$ converges to a $2k$ -cycle whose even-indexed terms are $(c_i, 0)$, $i = 1, \ldots, k$.*

(c) *If Q is chaotic on the invariant interval I (e.g., it a has snap-back repeller in I) then F is chaotic and has periodic points of all possible even periods. The even indexed terms are in $I \times \{0\}$ and the odd indexed terms are on the parabola (4).*

To see that the last case is in fact possible, consider a special case: $a = 0.465$, $d = 0.455$. In this case, direct computation shows that Q has a snap-back repeller in $I = [0.73, 0.95]$ and chaos obtains. See [6] for more details.

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